Lecture 24: Isomorphism Theorems.

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9:19 AM

In the previous lecture we defined

. Group factor G/N where $N \trianglelefteq G$

The 1^{st} isomorphism theorem

Let $\phi: G \longrightarrow H$ be a group homomorphism. Then

$$\overline{\Phi}: G/\ker\phi \longrightarrow \operatorname{Im}\phi, \quad \overline{\Phi}(g\ker\phi) := \phi(g)$$

is a group isomorphism.

Pf. We have already proved $\overline{\Phi}$ is a bijection. So it is enough to show it is a group homomorphism.

$$\overline{\Phi}(g_1\ker\phi \cdot g_2\ker\phi) = \overline{\Phi}((g_1g_2)\ker\phi)$$

$$= \phi(g_1g_2) = \phi(g_1)\phi(g_2)$$

$$= \overline{\Phi}(g_1\ker\phi)\ \overline{\Phi}(g_2\ker\phi). \quad \blacksquare$$

Exp. Let G be a cyclic group of order n. Then

$$G \simeq \mathbb{Z}_n.$$

Pf. $\left.\begin{array}{l} G \text{ is cyclic} \\ |G| = n \end{array}\right\} \Rightarrow \left\{\begin{array}{l} G = \langle g \rangle \\ o(g) = n \end{array}\right.$

Let $\mathbb{Z} \xrightarrow{\Phi} G$,

$m \longmapsto g^m$.

Then Φ is an epimorphism. $\ker\Phi = \{k\in\mathbb{Z} \mid g^k = e\}$

$= o(g)\mathbb{Z} = n\mathbb{Z}$

$\Rightarrow \mathbb{Z}/\ker\Phi \simeq \operatorname{Im}\Phi$

$\Rightarrow \mathbb{Z}/n\mathbb{Z} \simeq G$. ∎

<u>Exp.</u> Suppose G is cyclic and $\Phi: G \to H$ be a group homomorphism. Then $\operatorname{Im}(\Phi)$ is cyclic.

<u>Pf.</u> $G = \{g^i \mid i\in\mathbb{Z}\} \Rightarrow \operatorname{Im}\Phi = \{\Phi(g^i) \mid i\in\mathbb{Z}\}$

$= \{\Phi(g)^i \mid i\in\mathbb{Z}\}$

$= \langle \Phi(g) \rangle$. ∎

<u>Exp.</u> $\mathbb{R}^\times/\{\pm 1\} \simeq \mathbb{R}^+$

<u>Pf.</u> Let $\Phi: \mathbb{R}^\times \to \mathbb{R}^\times$, $\Phi(x) = x^2$.

$\Phi(xy) = (xy)^2 = x^2y^2 = \Phi(x)\Phi(y)$

$\Rightarrow \Phi$ is a group homomorphism.

$\Rightarrow \mathbb{R}^\times/\ker\Phi \simeq \operatorname{Im}\Phi$.

$\operatorname{Im}\phi = \mathbb{R}^+$ and $x \in \ker\phi \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$.

$\blacksquare$

Exp. $S_n/A_n \simeq \{\pm 1\} \simeq \mathbb{Z}_2$ if $n \geq 2$.

Pf. $\operatorname{sgn}: S_n \to \{\pm 1\}$ an epimorphism

$\ker \operatorname{sgn} = A_n \Rightarrow S_n/A_n \simeq \{\pm 1\} = \langle -1 \rangle \simeq \mathbb{Z}_2$.

$\blacksquare$

Exp. $\mathbb{Z}\times\mathbb{Z}/\langle(0,1)\rangle \simeq \mathbb{Z}$

Pf. $\phi: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $\phi(x,y) = x$

$$\Rightarrow \phi((x_1,y_1)+(x_2,y_2)) = \phi((x_1+x_2, y_1+y_2))$$
$$= x_1 + x_2$$
$$= \phi(x_1,y_1) + \phi(x_2,y_2)$$

And clearly ϕ is onto.

$(x,y) \in \ker\phi \Leftrightarrow \phi(x,y) = x = 0$

$\Rightarrow \ker\phi = \{0\}\times\mathbb{Z} = \langle(0,1)\rangle$.

So by the 1st isomorphism theorem we are done. $\blacksquare$

Exp. $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle \simeq \mathbb{Z}$

Pf. $\phi: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $\phi(x,y) = x - y$

It is easy to check that ϕ is a group homomorphism.

$\phi(x,0)=x \Rightarrow \phi$ is onto.

$(x,y)\in\ker\phi \Leftrightarrow x-y=0 \Leftrightarrow (x,y)=x\,(1,1).$

So $\ker\phi=\langle(1,1)\rangle$.

$\Rightarrow$ by 1^{st} isom. thm. we are done. $\blacksquare$

Exp. $\mathbb{Z}\times\mathbb{Z}/\langle(2,2)\rangle$ is NOT cyclic.

Pf. Suppose to the contrary that $\mathbb{Z}\times\mathbb{Z}/H = \langle(a,b)+H\rangle$

$\Rightarrow \forall (x,y)\in\mathbb{Z}\times\mathbb{Z},\ \exists n\in\mathbb{Z},$

$$(x,y)+H = n(a,b)+H$$

$$\Rightarrow (x,y)=n(a,b)+m(2,2)$$

$$\Rightarrow \begin{bmatrix} a & 2 \\ b & 2 \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$$

for any $x,y\in\mathbb{Z}$, $\exists\, m,n\in\mathbb{Z}$ s.t.

$\Rightarrow \begin{bmatrix} a & 2 \\ b & 2 \end{bmatrix}^{-1}$ exists and its entries are in $\mathbb{Z}$

$\Rightarrow 2a-2b=\pm 1$ which is a contradiction as the LHS is even and the RHS is odd. $\blacksquare$

Remark. The above argument can be modified to show

$$\mathbb{Z}\times\mathbb{Z}/\langle(c,d)\rangle \simeq \mathbb{Z} \Leftrightarrow \gcd(c,d)=1.$$

Exp. Let $Z(G)$ be the center of G. Prove that

if $G/_{Z(G)}$ is cyclic, then G is abelian.

In particular, $|G/_{Z(G)}|$ cannot be a prime number.

Pf. $G/_{Z(G)} = \langle g_0 Z(G) \rangle = \{ g_0^i Z(G) \mid i \in \mathbb{Z} \}$.

$g_1, g_2 \in G \Rightarrow \exists\, i_1 \in \mathbb{Z},\ z_1 \in Z(G)$

$\exists\, i_2 \in \mathbb{Z},\ z_2 \in Z(G)$ s.t.

$$g_1 = g_0^{i_1} z_1 \quad \text{and} \quad g_2 = g_0^{i_2} z_2$$

$$\Rightarrow g_1 g_2 = (g_0^{i_1} z_1)(g_0^{i_2} z_2)$$

$$= g_0^{i_1} g_0^{i_2} z_1 z_2$$

$$= g_0^{i_1+i_2} z_1 z_2$$

and
$$g_2 g_1 = g_0^{i_2+i_1} z_2 z_1$$

$$= g_0^{i_1+i_2} z_1 z_2.$$

$$\Rightarrow g_1 g_2 = g_2 g_1.$$

If a group is of prime p order, then $\forall\, g \in H \setminus \{e\}$,

$1 \neq o(g) \mid p \Rightarrow o(g) = p \Rightarrow H = \langle g \rangle$. ∎

$\Big\{\, 1 \neq o(g) \mid p \;\Rightarrow\; o(g) = p \;\Rightarrow\; H = \langle g \rangle.$ ∎

Exp. Let $H = \{\underset{e}{(1)}, \underset{a}{(1\ 2)(3\ 4)}, \underset{b}{(1\ 3)(2\ 4)}, \underset{c}{(1\ 4)(2\ 3)}\}$.

Prove that $H \trianglelefteq S_4$.

Pf. Subgroup.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$ab = (1\ 2)(3\ 4)(1\ 3)(2\ 4) = (1\ 4)(2\ 3) = c$

$ba = (ab)^{-1} = c^{-1} = c$

Normal. $\sigma (i_1\ \ i_2)(i_3\ \ i_4)\sigma^{-1}$

$= (\sigma(i_1)\ \ \sigma(i_2))(\sigma(i_3)\ \ \sigma(i_4)) \in H.$ ∎

Notice. $H \triangleleft A_4 \triangleleft S_4$

$\Rightarrow\; A_4/H \simeq \mathbb{Z}_3$ and $S_4/A_4 \simeq \mathbb{Z}_2$.

S_4 has no element of order $6 \Rightarrow S_4/H$ has no element of order 6.

<u>Exp.</u> Suppose $|G|=6$, $\exists\, a,b\in G$, $o(a)=2$ and $o(b)=3$.

$\Rightarrow$ either $G\simeq \mathbb{Z}_6$ or $G\simeq S_3$.

<u>Solution.</u> $o(b^2)=3 \Rightarrow a\notin \langle b\rangle$

$\Rightarrow \langle b\rangle \cap a\langle b\rangle = \emptyset$

$\Rightarrow G=\langle b\rangle \cup a\langle b\rangle$

$=\{e,b,b^2,a,ab,ab^2\}$

If $ab=ba \Rightarrow o(ab)=6 \Rightarrow G\simeq \mathbb{Z}_6$.

<u>$ab\neq ba$</u>. If $ba=e \Rightarrow b=a$ ※.

$ba=b^2 \Rightarrow b=a$ ※

$ba=a \Rightarrow b=e$ ※

So $\boxed{ba=ab^2}$ ... ∎