Lecture 25: Isomorphism theorems

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10:09 AM

The Second Isomorphism Theorem

$H \leq G$, $N \trianglelefteq G$ $\Rightarrow$ ① $HN \leq G$ and $H \cap N \trianglelefteq H$.

② $HN/N \simeq H/H\cap N$

Pf. Let $\pi : G \longrightarrow G/N$, $\pi(g) = gN$. Then we know

π is an onto group homomorphism and $\ker(\pi) = N$.

Let $\pi|_H : H \longrightarrow G/N$ be the restriction of π to H.

So $\pi|_H$ is a group homomorphism.

$\ker(\pi|_H) = \{h \in H \mid \pi(h) = N\} = H \cap N$.

$\text{Im}(\pi|_H) = \{hN \mid h \in H\} = HN/N$.

Notice that $\forall h \in H$, $hN = Nh \Rightarrow HN = NH$

$\Rightarrow HN \leq G \Rightarrow$ so it makes sense to write HN/N.

Hence, by the 1^{st} isomorphism theorem,

$\ker(\pi|_H) = H \cap N \trianglelefteq H$ and $\text{Im}(\pi|_H) = HN/N \leq G/N$

and

$$H/H\cap N \simeq HN/N. \quad \blacksquare$$

Exp. Suppose $H \leq G$ and $N \trianglelefteq G$. If $\gcd(|H|, [G:N])=1$, then $H \subseteq N$.

Pf. $H/_{H\cap N} \simeq HN/_N \leq G/_N$

$\Rightarrow$ By Lagrange's theorem, $\left|H/_{H\cap N}\right| \,\Big|\, |H|$

$$\overset{\parallel}{}$$

and $\left|HN/_N\right| \,\Big|\, \left|G/_N\right|$.

$\Rightarrow \left|H/_{H\cap N}\right| \,\Big|\, \gcd(|H|, [G:N]) = 1$

$\Rightarrow \left|H/_{H\cap N}\right| = 1 \Rightarrow H = H\cap N \Rightarrow H \subseteq N$. ■

The Third Isomorphism Theorem.

N and H are two normal subgroups of G, and $N \subseteq H$.

$\Rightarrow H/_N \trianglelefteq G/_N$ and $(G/N)/_{(H/N)} \simeq G/_H$.

Pf. Let $\theta : G/_N \longrightarrow G/_H$, $\theta(gN) := gH$.

well-defined

$$g_1 N = g_2 N \Rightarrow g_1^{-1} g_2 \in N$$

$$\Rightarrow g_1^{-1} g_2 \in H \quad (\text{as } N \subseteq H)$$

$$\Rightarrow g_1 H = g_2 H.$$

Homomorphism $\theta(g_1N \cdot g_2N) = \theta(g_1g_2N)$

$$= (g_1g_2)H$$

$$= g_1H \cdot g_2H$$

$$= \theta(g_1N) \cdot \theta(g_2N).$$

$\operatorname{Im}\theta = \{gH \mid g \in G\} = G/H.$

$gN \in \ker\theta \iff \theta(gN) = H \iff gH = H \iff g \in H$

$\iff gN \in H/N.$

So by the 1st isomorphism theorem

$$(G/N)/(H/N) \simeq G/H. \qquad \blacksquare$$

Exp. $nk\mathbb{Z} \leq n\mathbb{Z} \leq \mathbb{Z}$ and in an abelian group all the subgroups are normal. $\Rightarrow$

$$\mathbb{Z}_{nk}/n\mathbb{Z}_{nk} = (\mathbb{Z}/nk\mathbb{Z})/(n\mathbb{Z}/nk\mathbb{Z})$$

$$\simeq \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n. \qquad \blacksquare$$

Exp. $\mathbb{R}/\mathbb{Z} \simeq S^1.$

Pf. Let $\theta: \mathbb{R} \to S^1$, $\theta(x) = e^{2\pi i x}$.

$\Rightarrow \theta(x_1+x_2) = e^{2\pi i(x_1+x_2)} = e^{2\pi i x_1} \cdot e^{2\pi i x_2}$

$$= \theta(x_1)\,\theta(x_2)$$

and $|\theta(x)| = 1 \quad \forall x \in \mathbb{R}$.

- $\operatorname{Im}\theta = S^1$

- $x \in \ker\theta \iff e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$.

So by the 1st isomorphism we have $\mathbb{R}/\mathbb{Z} \simeq S^1$. ∎