Math 100a Fall 09 Final Exam

December 11, 2009

NAME:

Instructions:

Give full proofs on problems $2-7$. Generally, you may quote theorems we covered in class, which were covered in the textbook, or which you proved on your homework, in the course of your proof. However, *if the statement to be proved follows immediately from a theorem in the book or notes or a homework problem, you are expected to do the proof from scratch, using only more elementary facts and theorems in your proof.*

Problems 1 /20	
Problem 2 /10	
Problem 3 /10	
Problems 4 /10	
Problem 5 /10	
Problem 6 /15	
Problem 7 /10	
Total /85	

1. (20 pts, 4pts each) (short answer problems)
(a). Give an example of an infinite non-Abelian group (no proof needed.)

(b). Complete this statement. If $\phi : G_1 \to G_2$ is a homomorphism of groups, then the fundamental homomorphism theorem says that the following two groups are isomorphic:

(c). Let $\phi : G_1 \to G_2$ be a homomorphism of groups, where $|G_1| = 15$ and $|G_2| = 35$. Suppose that $\phi(G_1) \neq \{e\}$. Calculate $|\ker(\phi)|$. (no proof needed, but show your calculations.)

(d). State the *class equation* for a finite group G. Make sure you explain what the sum runs over.

(e). Suppose you know that G is a group of order 10, and that G has elements a and b where $a \neq b$ and $o(a) = o(b) = 2$. What familiar group must G be isomorphic to? Explain your answer in a few sentences.

2. (10 pts) In this problem, let m and n be positive integers, and let $G = \mathbb{Z}_m \times \mathbb{Z}_n$.

Suppose that $\gcd(m, n) > 1$. Prove that G is not cyclic.

3. (10 pts) Let $G = \mathbb{Z}_{20}^{\times}$, the units group of the integers modulo 20 under multiplication. Consider the cyclic subgroup $H = \langle [9] \rangle$ of G.

Show that the factor group G/H has order 4. We know that every group of order 4 is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Which of these groups is G/H isomorphic to? Justify your answer.

4. (10 pts) Let $\phi : G_1 \to G_2$ be a homomorphism of groups. Let $K = \ker(\phi) = \{x \in G_1 | \phi(x) = e\}$. Fix some $a \in G_1$ and suppose that $\phi(a) = b$.

Prove that given $g \in G_1$, $\phi(g) = b$ if and only if $g \in aK$.

(*Of course this is one of the properties of homomorphisms we proved, but I want you to prove it from scratch, using only more basic properties of homomorphisms.*)

5. (10 pts) Let G be a p-group for some prime p (that is, $|G| = p^i$ for some $i \geq 1$). Using the class equation, prove that the center $Z(G)$ has at least p elements.

6. (15 pts) Let $n \geq 3$ and consider the symmetric group S_n.

(a)(10 pts) Suppose that H is a subgroup of S_n with index 2, that is, $[S_n : H] = 2$. Suppose that $\sigma \in S_n$ is a k-cycle, where $k \geq 1$ is odd. Prove that $\sigma \in H$. (Hint: consider the factor group S_n/H.)

(b)(5 pts) Show that if H is a subgroup of S_4 with index 2, then $H = A_4$, the alternating subgroup. (Hint: use part (a).)

7. (10 pts) Let G be a group of order 12. Suppose that G has a subgroup H of order 3 which is *not* normal. Prove that $G \cong A_4$.

(Hint: Let S be the set of left cosets of H in G, and consider the usual action of G on S by left multiplication.)

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