QUIZ 3, VERSION A, MATH100B, WINTER 2021

1. (5 points) Suppose n is a positive odd integer. Prove that $f(x) = (x-2)(x-4)\cdots(x-2n)-1 \in \mathbb{Q}[x]$ is irreducible.

2. (5 points) Suppose $f, g \in \mathbb{Z}[x]$ are monic, p is prime, and $c_p : \mathbb{Z}[x] \to \mathbb{Z}_p[x]$ is the modulo-p residue map. Prove that if $\gcd(c_p(f), c_p(g)) = 1$ in $\mathbb{Z}_p[x]$, then $\gcd(f, g) = 1$ in $\mathbb{Q}[x]$.

3. Suppose D is a PID and $I = \langle p \rangle$ is a non-zero prime ideal of D.
   (a) (5 points) Prove that p is an irreducible element of D.

   (b) (3 points) Prove that I is a maximal ideal of D.

4. Suppose p is a prime, $a \in \mathbb{Z}_p^{\times}$, and $f(x) := x^p - x + a \in \mathbb{Z}_p[x]$. Suppose E is a field extension of $\mathbb{Z}_p$, and $\alpha \in E$ is a zero of $f(x)$. Notice that the characteristic of E is p.
   (a) (3 points) Prove that $x^p - x + a = (x-\alpha)\cdots(x-\alpha-(p-1))$ in $E[x]$.

   (b) (5 points) Prove that $x^p - x + a \in \mathbb{Z}_p[x]$ is irreducible.

   (c) (2 points) State the relevant results from the lectures or HW assignments and show that $\mathbb{Z}_p[\alpha]$ is a finite field of order p^p.

   (d) (2 points) Prove that $\prod_{a\in\mathbb{Z}_p^{\times}}(x^p - x + a)$ divides $x^{p^p} - x$.