QUIZ 1, MATH100C, SPRING 2021

1. Let $\zeta_n := e^{2\pi i/n}$.
   (a) (2 points) Prove that $\mathbb{Q}[\zeta_n]/\mathbb{Q}$ is Galois.

   (b) (2 points) Prove that $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}[\zeta_n])$ is abelian.

   (c) (2 points) Prove that $F/\mathbb{Q}$ is Galois for every $F \in \mathrm{Int}(\mathbb{Q}[\zeta_n]/\mathbb{Q})$.

   (d) (2 points) Prove that $\mathbb{Q}[\sqrt[3]{2}]$ is not a subfield of $\mathbb{Q}[\zeta_n]$ for any positive integer n.

2. Suppose F is a field of characteristic $p > 0$ and E/F is a field extension. Suppose $\gcd([E : F], p) = 1$.
   (a) (4 points) Prove that $m_{\alpha,F}(x)$ is separable in $F[x]$ for every $\alpha \in E$.

   (b) (2 points) Prove that E/F is a separable extension.

3. Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible and it has both a real and a non-real complex zero. Suppose $E \subseteq \mathbb{C}$ is a splitting field of f over $\mathbb{Q}$.
   (a) (2 points) Let $F := E \cap \mathbb{R}$. Prove that $[E : F] = 2$. (Hint: consider the complex conjugation $\tau : \mathbb{C} \to \mathbb{C}, \tau(z) := \overline{z}$ and argue that $\tau|_E \in \mathrm{Aut}_{\mathbb{Q}}(E)$.)

   (b) (4 points) Prove that $F/\mathbb{Q}$ is not a normal extension.

4. (10 points) Suppose $f \in \mathbb{Q}[x]$ is monic, irreducible and $\deg f = p$ is prime. Suppose $E \subseteq \mathbb{C}$ is a splitting field of f over $\mathbb{Q}$ and $\alpha \in E$ is a zero of f. Prove that there is $\theta \in \mathrm{Aut}_{\mathbb{Q}}(E)$ such that

$$f(x) = \prod_{i=0}^{p-1} (x - \theta^i(\alpha)).$$