QUIZ 3, MATH100C, SPRING 2021

1. (5 points) Suppose F is a field, $\overline{\mathrm{F}}$ is an algebraic closure of F, and $\alpha \in \overline{\mathrm{F}}$. Suppose $F[\alpha]/F$ is a Galois extension and $[F[\alpha] : F] = p$ where p is prime. Prove that $L[\alpha]/L$ is Galois and $[L[\alpha] : L]$ is either 1 or p, for every $L \in \mathrm{Int}(\overline{\mathrm{F}}/F)$.

2. Suppose $\overline{\mathbb{Q}} := \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$.
   (a) (3 point) Prove that $\overline{\mathbb{Q}}$ is algebraically closed.
   (b) (5 points) Suppose $\alpha_0 \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ and let $\Sigma_{\alpha_0} = \{E \in \mathrm{Int}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \alpha_0 \notin E\}$. Prove that Σ_{α_0} has a maximal element F with respect to inclusion.
   (c) (5 point) Suppose $F \in \Sigma_{\alpha_0}$ is a maximal element, and $E \in \mathrm{Int}(\overline{\mathbb{Q}}/F)$ and E/F is a finite Galois extension. Prove that $\mathrm{Aut}_F(E)$ is cyclic. (Hint. Argue that $F[\alpha_0] \subseteq K$ for every K in $\mathrm{Int}(E/F) \setminus \{F\}$ and think about Galois!)

3. (4 points) Suppose $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. Suppose $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$, and let $F := \mathrm{Fix}(\langle\sigma\rangle)$. Suppose $E \in \mathrm{Int}(\overline{\mathbb{Q}}/F)$ and E/F is a finite Galois extension. Prove that $\mathrm{Aut}_F(E) = \langle \sigma|_E \rangle$.

4. Suppose $\zeta_n := e^{\frac{2\pi i}{n}} \in \mathbb{C}$ and $K_n := \mathbb{Q}[\zeta_n] \cap \mathbb{R}$.
   (a) (4 points) Prove that $K_n/\mathbb{Q}$ is a Galois extension and $[K_n : \mathbb{Q}] = \frac{\phi(n)}{2}$ where $\phi(n)$ is the Euler ϕ-function.
   (b) (2 points) Prove that for every $\alpha \in K_n$ all the complex zeros of $m_{\alpha,\mathbb{Q}}$ are in $\mathbb{R}$.
   (c) (2 points) Suppose $\alpha \in K_n^{\times}$ and $\alpha^m \in \mathbb{Q}$ for some positive integer m. Prove that $\alpha^2 \in \mathbb{Q}$.