QUIZ 2, VERSION A — SOLUTIONS

Problem 1.

Part a. The set $\mathbb{Z} \setminus \{0\}$ is not a group under multiplication. Notice that $2 \in \mathbb{Z} \setminus \{0\}$ does not have a multiplicative inverse. To see this, suppose that $k \in \mathbb{Z} \setminus \{0\}$ such that $2k = 1$. Then we have that

$$1 = |2k| = 2|k| \geq 2 \cdot 1 > 1$$

which is a contradiction.

Part b. The subset $\mathbb{Q} \setminus \{0\} \subset \mathbb{Q}$ is not a subgroup under addition since it does not contain the neutral element (0).

Part c. The set $\mathbb{Z}_{15} \setminus \{[0]_{15}\}$ is not a group under multiplication. To see this, notice that $[3]_{15}$ does not have a multiplicative inverse since $\gcd(3, 15) = 3 \neq 1$.

Part d. We have that

$$f([2]_7) = [2]_7^3 = [2^3]_7 = [8]_7 = [1]_7$$

and $[1]_7$ is the identity element of $\mathbb{Z}_7^\times$. Thus $[2]_7 \in \ker f$.

Part e. The group S_3 is not abelian since S_n is not abelian for any $n \geq 3$.

Problem 2. First we do the Euclidean algorithm

$$\begin{aligned} 20 &= 1 \cdot 13 + 7 \\ 13 &= 1 \cdot 7 + 6 \\ 7 &= 1 \cdot 6 + 1 \\ 6 &= 6 \cdot 1. \end{aligned}$$

Now we calculate $x, y \in \mathbb{Z}$ such that $20x + 13y = 1$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 20 \\ 13 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -13 & 20 \end{pmatrix} \begin{pmatrix} 20 \\ 13 \end{pmatrix}.$$

Therefore $2 \cdot 20 + (-3) \cdot 13 = 1$ so $[13]_{20}^{-1} = [-3]_{20} = [17]_{20}$.

Problem 3. First notice that since for any $g \in G$, we have that $g^2 = e$, we have that $g = g^{-1}$. Therefore, for any $x, y \in G$,

$$xy = (xy)^{-1} = y^{-1}x^{-1} = yx$$

so G is abelian.

Problem 4.

Part a. Suppose that $x \in \mathbb{Z}_n$. Then we have that

$$\tau \circ \sigma^2 \circ \tau(x) = \tau(\sigma^2(\tau(x))) = \tau(\sigma^2(-x)) = \tau(-x + 2) = -(-x + 2) = x - 2.$$

Thus $\tau \circ \sigma^2 \circ \tau = \sigma^{-2}$.

Part b. We have that

$$\sigma^3 \in C_{D_10}(\tau) \iff \tau\sigma^3 = \sigma^3\tau = \tau\sigma^3\tau^{-1} = \sigma^3.$$

Thus we only have to show that $\tau\sigma^3\tau^{-1} \neq \sigma^3$. Notice that

$$\tau \circ \sigma^3 \circ \tau^{-1}([0]_{10}) = \tau \circ \sigma^3([0]_{10}) = \tau([3]_{10}) = [-3]_{10} = [7]_{10}.$$

Since $\sigma^3([0]_{10}) = [3]_{10} \neq [7]_{10}$ we have the desired result.

Part e. First notice that

$$\sigma^{-4} \circ \gamma([0]_{10}) = \sigma^{-4}([4]_{10}) = [0]_{10}.$$

Since

$$\sigma^{-4} \circ \gamma([1]_{10}) = [-1]_{10}$$

we have that $\sigma^{-4} \circ \gamma = \tau$. Thus $\gamma = \sigma^4 \circ \tau$, but that's not quite the right form we want. However, notice that for any $x \in \mathbb{Z}_{10}$, we have that

$$\sigma^4 \circ \tau(x) = \sigma^4(-x) = -x + 4.$$

This is the same as $\tau \circ \sigma^{-4}$, so $\gamma = \tau \circ \sigma^4$. Alternatively, you can notice that $\tau \circ \sigma^n \circ \tau^{-1} = \sigma^{-n}$ for any $n \in \mathbb{Z}$. (Look at the 7/19 office hour to see a more thorough explanation. Essentially how one can use $\tau\sigma\tau^{-1} = \sigma^{-1}$ to understand the multiplication table of the dihedral group.)