QUIZ 3, VERSION A — SOLUTIONS

Problem 1.

Part a. This statement is true. Notice that $[2]_5^4 = [16]_5 = [1]_5$ so $o([2]_5) \mid 4$. Since $[2]_5 \neq [1]_5$ and $[2]_5^2 = [4]_5 \neq [1]_5$ we see that $o([2]_5) = 4$.

Part b. This statement is true. Consider the natural log map $\ln : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$. Notice that for any two positive $x, y \in \mathbb{R}$, we have that

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

so ln is a group homomorphism. Furthermore, we know that ln is bijective and so it is a group isomorphism.

Part c. This statement is false. Consider $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7) \in S_7$. Since the two cycles in the above decomposition are disjoint, we have

$$o(\sigma) = \mathrm{lcm}(o((1\ 2\ 3)), o((4\ 5\ 6\ 7)) = \mathrm{lcm}(3, 4) = 12 > 7.$$

Part d. This statement is false. Consider $\sigma = (1\ 2)$ and $\tau = (2\ 3)$, both in S_3. Clearly $\sigma^2 = \tau^2 = \mathrm{id}$. However, $\sigma\tau = (1\ 2\ 3)$, which has order 3.

Problem 2.

Part a. Suppose $(x, y) \in G \times G$. Then

$$(x, y)^{30} = (x^{30}, y^{30}) = (e_G, e_G)$$

since for every $g \in G$, $g^{30} = e_G$ as it is a cyclic group of order 30.

Part b. Suppose $(x, y) \in G$. By the previous part, $o(x, y) \mid 30$. In particular, $o(x, y) \leq 30$. Therefore

$$|\langle (x, y) \rangle| = o(x, y) \leq 30 < 60 = |G \times G|.$$

Therefore no element can generate $G \times G$, and so the direct product is not cyclic.

Part c. From lecture, we have that

$$o(g^k) = \frac{o(g)}{\gcd(o(g), k)}.$$

Therefore

$$\gcd(o(g), k) = \frac{o(g)}{o(g^k)} = \frac{30}{15} = 2.$$

Part d. By the previous part, the elements of G with order 15 are those $g^k \in G$ where $0 \leq k < 30$ and $\gcd(k, 30) = 2$. Namely, k needs to lie in $\{2, 4, 8, 14, 16, 22, 26, 28\}$. Therefore there are 8 elements of order 15 in G.

(Alternatively, $\gcd(k, 30) = 2$ implies that $k = 2l$, $\gcd(l, 15) = 1$, and $1 \leq 2l \leq 30$. Hence the number such l's is equal to $\phi(15) = \phi(3)\phi(5) = 2 \times 4 = 8$.)

Part e. From lecture, we know that there is a bijection between subgroups of a finite cyclic group and positive divisors of its order. Hence the number of subgroups of G is equal to the number of positive divisors of 30. The set of positive divisors of 30 is $\{1, 2, 3, 5, 6, 10, 15, 30\}$. Therefore G has 8 subgroups.

Problem 3.

Part a. The cycle decomposition of σ is

$$\sigma = (1\ 10\ 5)(2\ 9\ 6\ 3\ 7)(4\ 8).$$

Part b. From lecture, we know that the order of a permutation σ is equal to the least common multiple of the lengths of the cycles in its cycle decomposition. Hence

$$o(\sigma) = \operatorname{lcm}(3, 5, 2) = 30.$$

Therefore $|\langle\sigma\rangle| = o(\sigma) = 30$.

Part c. Since the order of σ is 30, notice that

$$\sigma^{59}\sigma = \sigma^{60} = \mathrm{id}.$$

Therefore $\sigma^{59} = \sigma^{-1}$. Thus

$$\sigma^{59} = (5\ 10\ 1)(7\ 3\ 6\ 9\ 2)(8\ 4).$$

Notice that to invert σ, you just reverse all of the cycles in the disjoint cycle decomposition.

Part d. From lecture, we know that we can write every n-cycle as the product of $n - 1$ transpositions. Therefore, σ can be written as a product of

$$(3 - 1) + (5 - 1) + (2 - 1) = 7$$

transpositions. Hence σ is odd.

(Alternatively, sign of an m-cycle is $(-1)^{m-1}$, and so

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(1\ 10\ 5)\operatorname{sgn}(2\ 9\ 6\ 3\ 7)\operatorname{sgn}(4\ 8) = (1)(1)(-1) = -1.$$

This means that σ is odd.)