Homework 1

Friday, April 6, 2018 10:32 AM

1. Suppose $R_1, \ldots, R_n$ are rings. Prove that $R_1, \ldots, R_n$ are unital if and only if $R_1 \times \cdots \times R_n$ is unital.

2. Suppose R is a unital ring. An element x of R is called a <u>unit</u> if it has a multiplicative inverse; that means $\exists\, x' \in R$ such that $x x' = x' x = 1_R$.

   Let $U(R)$ be the set of all the units of R.

   ⓐ Prove that $U(R)$ is closed under multiplication.

   ⓑ Prove that $(U(R), \cdot)$ is a group.

   ⓒ Suppose R_i's are unital rings. Prove that
   $$U(R_1 \times \cdots \times R_n) = U(R_1) \times \cdots \times U(R_n).$$

   ⓓ Find $U(\mathbb{Z} \times \mathbb{Q})$.

3. Show that $\{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$ is ring.

4. As in problem 3 one can show $F = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ is a ring. Show that $U(F) = F \setminus \{0\}$; that means any non-zero element is a unit.

Homework 1

Friday, April 6, 2018 10:34 AM

5. For a ring R, let $R[x] = \{a_0 + a_1 x + \cdots + a_n x^n \mid n \in \mathbb{Z}^{\geq 0}, a_0, \ldots, a_n \in R\}$ be the ring of polynomials with coefficients in R and indeterminant x. We add and multiply polynomials as usual.

   (a) Show that $U(\mathbb{Z}[x]) = \{\pm 1\}$.

   (b) Show that $2x+1 \in U(\mathbb{Z}_8[x])$.

6. Suppose A is a ring with unity 1. Suppose there is $a_0 \in A$ such that $a_0^2 = 1$. Let $B := \{a_0 r a_0 \mid r \in A\}$. Prove that B is a subring of A.