Homework 2

Thursday, April 12, 2018 11:59 PM

1. (a) Find all the solutions of x^2-x-2 in $\mathbb{Z}_{17}$.

   (b) Does x^2-x-2 have only two zeros in $\mathbb{Z}_{18}$?

2. Find the characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6$ and $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9$.

   (Justify your answer.).

3. (a) Show that $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}\}$ is a field.

   (b) Similarly one can show that $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Z}\}$

   is a ring. Prove that $\mathbb{Q}[\sqrt{2}]$ is the field of fractions

   of $\mathbb{Z}[\sqrt{2}]$ (up to an isomorphism).

4. Let $f: \mathbb{Z}[\sqrt{2}] \longrightarrow \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| a,b \in \mathbb{Z} \right\}$,

   $f(a+\sqrt{2}\,b) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism of rings.

   (You do not need to show that the codomain is a subring

   of $M_2(\mathbb{Z})$.)

5. Suppose A is a unital commutative ring of characteristic $p>0$,

   where p is prime. Prove that, for any $x,y \in A$, $(x+y)^p = x^p + y^p$.

   (Hint. You are allowed to use binomial expansion without proof.

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$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ where $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$, and

$\binom{n}{i} \in \mathbb{Z}$.

Prove $p \mid \binom{p}{i}$ if $0 < i < p$ and p is prime.

Deduce the claim.)

6 (a) Find a zero-divisor in $\mathbb{Z}_5[i] = \{a+bi \mid a, b \in \mathbb{Z}_5\}$

where $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$.

(You do not need to show it is a ring.)

(b) Show that x^2+1 has no zero in $\mathbb{Z}_7$.

(c) Show that, if either $a \neq 0$ or $b \neq 0$ in $\mathbb{Z}_7$, then $a^2+b^2 \neq 0$ in $\mathbb{Z}_7$.

(d) Show that $\mathbb{Z}_7[i] = \{a+bi \mid a, b \in \mathbb{Z}_7\}$ is a field.

(Hint. (c) if $a \neq 0$, then $a^2+b^2 = a^2\left(1+\left(\frac{b}{a}\right)^2\right)$; use (b).

(d) It is enough to show $\mathbb{Z}_7[i]$ is an integral domain. (why?)

$(a+bi)(c+di) = 0 \Rightarrow (a+bi)(a-bi)(c-di) = 0$

$\Rightarrow (a^2+b^2)(c^2+d^2) = 0$ in $\mathbb{Z}_7$ use (c) to

show either $a+bi = 0$ or $c+di = 0$.)