Homework 3

Thursday, April 19, 2018 9:08 AM

1. (a) Show that $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$ where $\omega = \frac{-1+\sqrt{-3}}{2}$.

   (b) Show that the field of fractions of $\mathbb{Z}[\omega]$ is $\mathbb{Q}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Q}\}$.

(Hint. Use $\omega^2 + \omega + 1 = 0$; and compute $(a+b\omega)(a+b\bar{\omega})$ where $\bar{\omega} = \frac{-1-\sqrt{-3}}{2}$. (Notice $\omega + \bar{\omega} = -1$ and $\omega\bar{\omega} = 1$.))

2. (a) Suppose R is a unital commutative ring. Prove that $\langle u \rangle = R$ if and only if $u \in U(R)$.

   (b) Suppose D is an integral domain. Prove that

$$\langle a \rangle = \langle b \rangle \Leftrightarrow \exists u \in U(D) \text{ s.t. } a = bu.$$

(Hint. For (b), $\left.\begin{array}{l} a \overset{?}{=} b\,u \\ b \overset{?}{=} a\,u' \end{array}\right\} \Rightarrow$ try to deduce $uu' = 1$.)

3. Suppose R_1 and R_2 are unital commutative rings, and $I \trianglelefteq R_1 \times R_2$. Prove that $I = I_1 \times I_2$ for some $I_1 \trianglelefteq R_1$ and $I_2 \trianglelefteq R_2$.

(Hint. Suppose $(x_1, x_2) \in I$. Then $(x_1, x_2) \cdot (1_{R_1}, 0_{R_2}) \in I$.)

4. Prove that $\langle 2, x \rangle \trianglelefteq \mathbb{Z}[x]$ is not a principal ideal.

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(Hint. Suppose to the contrary $\langle f(x)\rangle = \langle 2, x\rangle$. So

$$f(x)\cdot h(x) \overset{(I)}{=} 2 \text{ and } f(x)\cdot g(x) \overset{(II)}{=} x \text{ for some } h(x), g(x) \in \mathbb{Z}[x].$$

- What can you say about the degree of f using (I)?
- Show $f(x)$ is either ± 1 or ± 2 using (I).
- Using (II) deduce $f(x)$ should be ± 1 and get a contradiction.)

5. (a) Find the remainder of 102459087 divided by 9.

   (b) Find the remainder of 102459087 divided by 11.

   (c) Compute $2/3$ in $\mathbb{Z}_{11}$, $2/7$ in $\mathbb{Z}_{19}$, and $2/9$ in $\mathbb{Z}_{23}$.

(Justify your answers; and do not use long division to find the remainders.)

6. Let $f : \mathbb{Z}[i] \to \mathbb{Z}_5$, $f(a+bi) = \bar{a} \oplus 2\bar{b}$ where $\bar{a}$ is the remainder of a divided by 5 and $\bar{b}$ is the remainder of b divided by 5.

   (a) Prove that f is a ring homomorphism.

   (b) Show that $\langle -2+i\rangle \subseteq \ker f$.