Homework 9

Wednesday, May 30, 2018 8:32 AM

1. (a) Prove that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3[x]$.

(b) Prove that $\mathbb{Z}_3[x]/\langle x^3 - x + 1\rangle$ is a field.

(c) Prove that there is a field F that has 27 elements, and it has a zero α of $x^3 - x + 1$.

(Hint. For part (c), use part (b) and long division.)

2. Suppose $f(x) = x^5 - 6x^4 + 30x + 12$.

(a) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) Suppose $\alpha \in \mathbb{C}$ is a zero of f. Prove that

$$\{a_0 + a_1\alpha + \cdots + a_4\alpha^4 \mid a_0, \ldots, a_4 \in \mathbb{Q}\}$$

is a field.

(c) Prove that $1, \alpha, \ldots, \alpha^4$ are linearly independent over $\mathbb{Q}$; that means: if $a_0 + a_1\alpha + \cdots + a_4\alpha^4 = 0$ for some $a_i \in \mathbb{Q}$, then $a_0 = a_1 = \cdots = a_4 = 0$.

3. Suppose p is an odd prime. Prove that $\underbrace{x^{p-1} - x^{p-2} + \cdots + x^2 - x + 1}_{f(x).}$ is irreducible in $\mathbb{Q}[x]$. (Hint. Consider $f(-x)$.)

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Wednesday, May 30, 2018 8:49 AM

4. Let $\alpha = \sqrt{1+\sqrt{3}}$.

(a) Prove that $x^4 - 2x^2 - 2$ is a minimal polynomial of α over $\mathbb{Q}$.

(b) $\{a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Q}\}$ is a field.

5. Show that there is a finite field of order 25.

(Hint. Find a degree 2 irreducible polynomial in $\mathbb{Z}_5[x]$.)