Lecture 06: Field of fractions

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In the previous lecture we stated the following theorem:

<u>Theorem</u>. Let D be an integral domain. Then there are a field $Q(D)$ and a ring homomorphism $\theta : D \to Q(D)$ s.t. (1) θ is injective, (2) any element of $Q(D)$ is of the form $\theta(a)\,\theta(b)^{-1}$ for some $a \in D$, $b \in D \setminus \{0\}$. A field with these conditions satisfies the following property: if $\psi : D \hookrightarrow F$ is an injective ring homomorphism, then there is an injective hom. $\widetilde{\psi} : Q(D) \to F$ s.t. $\widetilde{\psi}(\theta(d)) = \psi(d)$ for any $d \in D$. In particular a field with conditions (1) and (2) is unique up to an isomorphism; and it is called the field of fractions of D.

- We follow the method of construction of $\mathbb{Q}$ from $\mathbb{Z}$;

Informally elements of $Q(D)$ are of the form numerator/denom. and denom. cannot be zero. But there is subtlety,

$a/b = ar/br$ for any $r \in D \setminus \{0\}$. And so we start with $D \times (D \setminus \{0\})$ and then partition it in a way that (a_1, b_1) and (a_2, b_2) are in the same subset precisely when "$a_1/b_1 = a_2/b_2$"; that should mean $a_1 b_2 = a_2 b_1$. And so we define

$$[(a,b)] = \{(a', b') \in D \times (D \setminus \{0\}) \mid ab' = a'b\}$$

<u>Lemma</u>. $\{[(a,b)] \mid (a,b) \in D \times (D \setminus \{0\})\}$ is a partition

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of $D\times(D\setminus\{0\})$.

<u>Pf.</u> ∘ $(a,b)\in[(a,b)]$ as $ab=ab$. And so

$$\bigcup_{(a,b)\in D\times(D\setminus\{0\})}[(a,b)]=D\times(D\setminus\{0\}).$$

∘ $(c,d)\in[(a,b)]\iff[(c,d)]=[(a,b)]$

($\Leftarrow$) $(c,d)\in[(c,d)]=[(a,b)]$.

($\Rightarrow$) $(c,d)\in[(a,b)]\Rightarrow ad=bc$

$(x,y)\in[(c,d)]\Rightarrow xd=yc\Rightarrow bx\cancel{d}=y\,bc=y\,a\cancel{d}$

$\boxed{\text{Cancellation property}}\Rightarrow bx=ya\Rightarrow(x,y)\in[(a,b)]$.

And so $[(c,d)]\subseteq[(a,b)]$.

$(x,y)\in[(a,b)]\Rightarrow ay=bx\Rightarrow ayc=bcx=ad\,x$

$\boxed{\text{Cancellation property}}\Rightarrow yc=dx\Rightarrow(x,y)\in[(c,d)]$

And so $[(a,b)]\subseteq[(c,d)]$.

• $(c,d)\in[(a_1,b_1)]\cap[(a_2,b_2)]\Rightarrow$,

$$[(a_1,b_1)]=[(c,d)]=[(a_2,b_2)].$$

and claim follows. ∎

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<u>Corollary</u>. $[(a_1, b_1)] = [(a_2, b_2)] \Leftrightarrow a_1 b_2 = a_2 b_1$.

<u>Pf</u>. We have proved that

$$[(a_1, b_1)] = [(a_2, b_2)] \Leftrightarrow (a_1, b_1) \in [(a_2, b_2)]$$

$$\Leftrightarrow a_1 b_2 = b_1 a_2. \quad \blacksquare$$

Next we make $Q(D) := \{ [(a,b)] \mid (a,b) \in D \times (D \setminus \{0\}) \}$ into a field.

<u>Lemma</u>. The following are well-defined binary operations on $Q(D)$.

$$[(a,b)] + [(c,d)] = [(ad+bc, bd)] \text{ and } [(a,b)] \cdot [(c,d)] = [(ac, bd)].$$

(Of course, this definition is inspired by the addition and multiplication of fractions in $\mathbb{Q}$)

<u>Pf</u>.
$$\left.\begin{array}{l} [(a_1,b_1)] = [(a_2,b_2)] \\ [(c_1,d_1)] = [(c_2,d_2)] \end{array}\right\} \overset{?}{\Longrightarrow} [(a_1d_1+b_1c_1, b_1d_1)] = [(a_2d_2+b_2c_2, b_2d_2)]$$

We know $a_1 b_2 = a_2 b_1$ and $c_1 d_2 = d_1 c_2$; and we'd like to show

$$(a_1d_1+b_1c_1)\, b_2 d_2 \overset{?}{=} (a_2 d_2 + b_2 c_2)\, b_1 d_1.$$

$$(a_1d_1+b_1c_1) b_2 d_2 = a_1 d_1 b_2 d_2 + b_1 c_1 b_2 d_2 = a_2 d_1 b_1 d_2 + b_1 c_2 b_2 d_1 = (a_2 d_2 + c_2 b_2)\, b_1 d_1.$$

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Similarly we have to show $[(a_1,b_1)]=[(a_2,b_2)]$ and $[(c_1,d_1)]=[(c_2,d_2)]$ implies $[(a_1c_1,b_1d_1)]=[(a_2c_2,b_2d_2)]$. We know $a_1b_2=a_2b_1$ and $c_1d_2=c_2d_1$; and we'd like to show $a_1c_1b_2d_2=b_1d_1a_2c_2$.

$a_1\,c_1\,b_2\,d_2 = a_2\,c_2\,b_1\,d_1$. ∎

Lemma. For any $a\in D\setminus\{0\}$, $[(0,1)]=[(0,a)]$; and it is the neutral element of $+$ in $Q(D)$.

Pf. • $0\cdot a=1\cdot 0 \Rightarrow [(0,1)]=[(0,a)]$.

• $[(0,1)]+[(c,d)]=[(0\cdot d+1\cdot c,\ 1\cdot d)]=[(c,d)]$. Similarly $[(c,d)]+[(0,1)]=[(c,d)]$. ∎

Lemma. $(Q(D),+,\cdot)$ is a commutative ring.

Pf. One has to check associativity and distribution. These are rather easy and I leave them as exercise.

Lemma. For any $a\in D\setminus\{0\}$, $[(a,a)]=[(1,1)]$ and it is the unity of $Q(D)$; and $[(0,1)]\neq[(1,1)]$.

Pf. • $a\cdot 1=1\cdot a \Rightarrow [(a,a)]=[(1,1)]$

• $[(a,b)]\cdot[(1,1)]=[(a\cdot 1,b\cdot 1)]=[(a,b)]$.

• $[(0,1)]=[(1,1)]$ implies $0\cdot 1=1\cdot 1 \Rightarrow 0=1$ which is a contr. as D is an integral domain. ∎

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Lemma. $Q(D)$ is a field.

Pf. Since we have already proved $Q(D)$ is a non-zero unital commutative ring, it is enough to show any non-zero element has an inverse.

$[(a,b)] \neq [(0,1)] \Rightarrow a \in D \setminus \{0\} \Rightarrow [(b,a)] \in Q(D)$ and

$$[(a,b)]\,[(b,a)] = [(ab,ab)] = [(1,1)]. \qquad \blacksquare$$

Lemma. $\theta : D \to Q(D)$, $\theta(d) = [(d,1)]$ is an injective ring homomorphi. and any element of $Q(D)$ is of the form $\theta(a)\,\theta(b)^{-1}$ for some $a \in D$, $b \in D \setminus \{0\}$.

(we will continue next time)