Lecture 07: Field of fractions

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We were proving the following Theorem:

Theorem. Let D be an integral domain. Then there are a field $Q(D)$ and a ring homomorphism $\theta: D \to Q(D)$ s.t. (1) $\underset{\theta(1)=1}{\theta}$ is injective, (2) any element of $Q(D)$ is of the form $\theta(a)\theta(b)^{-1}$ for some $a \in D$, $b \in D\setminus\{0\}$. A field with these conditions satisfies the following property: if $\underset{\psi(1)=1}{\psi}: D \hookrightarrow F$ is an injective ring homomorphism, then there is an injective hom. $\tilde{\psi}: Q(D) \to F$ s.t. $\tilde{\psi}(\theta(d)) = \psi(d)$ for any $d \in D$. In particular a field with conditions (1) and (2) is unique up to an isomorphism; and it is called the field of fractions of D.

We have defined $Q(D) := \{[(a,b)] \mid (a,b) \in D \times (D\setminus\{0\})\}$, and defined $+, \cdot$ following the model of $\mathbb{Q}$, and showed:

. $(Q(D), +, \cdot)$ is a field. We then showed:

Lemma. $\theta: D \to Q(D)$, $\theta(a) = [(a,1)]$ satisfies (1) and (2).

pf. $\theta(a+b) = [(a,1)] + [(b,1)] = [(a \cdot 1 + 1 \cdot b, 1 \cdot 1)]$
$= [(a+b, 1)] = \theta(a+b)$

. $\theta(ab) = [(ab,1)] = [(a,1)][(b,1)] = \theta(a)\theta(b)$.

. $\theta(a_1) = \theta(a_2) \Rightarrow [(a_1,1)] = [(a_2,1)] \Rightarrow a_1 \cdot 1 = a_2 \cdot 1 \Rightarrow a_1 = a_2$.

. $[(a,b)] = [(a,1)][(1,b)] = [(a,1)][(b,1)]^{-1} = \theta(a)\theta(b)^{-1}$ ■

Lemma. Suppose Q is a field and $\theta: D \to Q$ satisfies condition (1) and (2). Then for any injective ring homomorphism $\psi: D \to F$

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$\exists\ \tilde{\psi} : Q \to F$ s.t. $\tilde{\psi}$ is injective and $\tilde{\psi}(\theta(d)) = \psi(d)$.

<u>Pf</u>. Let $\tilde{\psi}(\theta(a)\theta(b)^{-1}) := \psi(a)\psi(b)^{-1}$ for $a \in D$, $b \in D \setminus \{0\}$.

<u>$\tilde{\psi}$ is well-defined.</u> we have to show

$$\theta(a_1)\theta(b_1)^{-1} = \theta(a_2)\theta(b_3)^{-1} \quad \text{implies} \quad \psi(a_1)\psi(b_1)^{-1} \overset{?}{=} \psi(a_2)\psi(b_2)^{-1}.$$

$$\theta(a_1)\theta(b_1)^{-1} = \theta(a_2)\theta(b_2)^{-1} \Rightarrow \theta(a_1)\theta(b_2) = \theta(b_1)\theta(a_2)$$

$$\Rightarrow \theta(a_1 b_2) = \theta(b_1 a_2) \Rightarrow a_1 b_2 = b_1 a_2 \text{ as } \theta \text{ is injective}$$

$$\Rightarrow \psi(a_1 b_2) = \psi(b_1 a_2) \Rightarrow \psi(a_1)\psi(b_2) = \psi(b_1)\psi(a_2)$$

$$\Rightarrow \psi(a_1)\psi(b_1)^{-1} = \psi(a_2)\psi(b_2)^{-1}.$$

<u>$\tilde{\psi}$ preserves addition</u>.

$$\tilde{\psi}(\theta(a_1)\theta(b_1)^{-1} + \theta(a_2)\theta(b_2)^{-1})$$

$$= \tilde{\psi}\left(\left(\theta(a_1)\theta(b_2) + \theta(a_2)\theta(b_1)\right)\left(\theta(b_1)^{-1}\theta(b_2)^{-1}\right)\right)$$

$$= \tilde{\psi}\left(\theta(a_1 b_2 + a_2 b_1)\,\theta(b_1 b_2)^{-1}\right) = \psi(a_1 b_2 + a_2 b_1)\,\psi(b_1 b_2)^{-1}$$

$$= \left(\psi(a_1)\psi(b_2) + \psi(a_2)\psi(b_1)\right)\psi(b_1)^{-1}\psi(b_2)^{-1}$$

$$= \psi(a_1)\psi(b_1)^{-1} + \psi(a_2)\psi(b_2)^{-1} = \tilde{\psi}(\theta(a_1)\theta(b_1)^{-1}) + \tilde{\psi}(\theta(a_2)\theta(b_2)^{-1}).$$

Similarly one can show that it preserves multiplication.

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$\tilde{\psi}$ is injective

From group theory, we know it is enough to show

$$\ker(\tilde{\psi}) := \{q \in Q \mid \tilde{\psi}(q) = 0\} = 0.$$

Suppose $\tilde{\psi}(\theta(a)\theta(b)^{-1}) = 0$. So $\psi(a)\psi(b)^{-1} = 0$; therefore $\psi(a) = 0$, which implies $a = 0$. Thus $\theta(a)\theta(b)^{-1} = 0$. ∎

Lemma. Suppose Q_1 and Q_2 satisfy (1) and (2). Then $Q_1 \simeq Q_2$, which means there is an isomorphism from Q_1 to Q_2.

Pf. Suppose $\theta_1 : D \to Q_1$ and $\theta_2 : D \to Q_2$ are the injective ring hom. that give us (1) and (2). By the above lemma $\exists\ \tilde{\theta}_1 : Q_2 \to Q_1$, $\tilde{\theta}_1(\theta_2(a)\theta_2(b)^{-1}) = \theta_1(a)\theta_1(b)^{-1}$

$\exists\ \tilde{\theta}_2 : Q_1 \to Q_2$, $\tilde{\theta}_2(\theta_1(a)\theta_1(b)^{-1}) = \theta_2(a)\theta_2(b)^{-1}$.

Hence $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are inverse of each other. Therefore they are isomorphisms. ∎

Lecture 07: A remark about kernel; Field of fractions of a field.

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Recall from group theory. Suppose A and B are additive groups, $f: A \to B$ is an additive group homomorphism. Then

$$\ker(f) = \{a \in A \mid f(a) = 0\} \text{ is called the kernel of } f.$$

$\ker(f)$ is a (normal) subgroup of A. And f is injective if and only if $\ker(f) = 0$.

Pf of the last part $(\Rightarrow)$ $a \in \ker(f) \Rightarrow f(a) = 0 = f(0)$

$$\Rightarrow a = 0.$$

$(\Leftarrow)$ $f(a_1) = f(a_2) \Rightarrow f(a_1 - a_2) = 0 \Rightarrow a_1 - a_2 \in \ker(f) = 0$

$$\Rightarrow a_1 - a_2 = 0 \Rightarrow a_1 = a_2. \quad \blacksquare$$

Ex. Suppose F is a field. Then F is the field of fractions of F.

Pf. Let $\theta: F \to F$, $\theta(x) = x$. Then ⓪ F is a field, ① θ is an injective ring homomorphism, $\theta(1) = 1$, ② Any element of F is of the form $a = \theta(a)\,\theta(1)^{-1}$. And so F is the field of fractions of F. $\blacksquare$

Lecture 07: Field of fractions of the Gaussian integers

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<u>Ex.</u> Show that $\mathbb{Q}[i] := \{a+bi \mid a,b \in \mathbb{Q}\}$ is the field of fractions of $\mathbb{Z}[i]$.

<u>Pf.</u> Let $\theta: \mathbb{Z}[i] \to \mathbb{Q}[i]$, $\theta(x) = x$. We have to show

(0) $\mathbb{Q}[i]$ is a field, (1) θ is an injective ring hom, $\theta(1)=1$

(2) Any element of $\mathbb{Q}[i]$ is of the form $\theta(a+bi)\,\theta(c+di)$ for some $a,b,c,d \in \mathbb{Z}$.

<u>(0)</u>. Using a subring criterion one can show $\mathbb{Q}[i]$ is a subring of $\mathbb{C}$. So it is enough to show that any non-zero element of $\mathbb{Q}[i]$ has an inverse. Suppose $a+ib \neq 0$, $a,b \in \mathbb{Q}$. Then

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i.$$ Since $a+ib \neq 0$, $a^2+b^2 \neq 0$.

Since $a,b \in \mathbb{Q}$, $\frac{a}{a^2+b^2}$ and $\frac{b}{a^2+b^2}$ are in $\mathbb{Q}$. Hence

$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i \in \mathbb{Q}[i].$$

<u>(1)</u> is clear.

<u>(2)</u> Any element of $\mathbb{Q}[i]$ is of the form $\frac{a}{b} + i\frac{c}{d}$, for some $a,b,c,d \in \mathbb{Q}$ and $bd \neq 0$. $\frac{a}{b} + i\frac{c}{d} = \frac{ad+icb}{bd}$ and $ad+icb, bd \in \mathbb{Z}[i]$