Homework 2

Saturday, January 19, 2019 10:39 AM

1. Let $f: \mathbb{Z}[\sqrt{2}] \longrightarrow \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a,b \in \mathbb{Z} \right\}$,

$f(a+\sqrt{2}\, b) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism of rings.

(You do not need to show that the codomain is a subring of $M_2(\mathbb{Z})$.)

2. Suppose A is a unital commutative ring of characteristic $p>0$, where p is prime. Prove that, for any $x,y \in A$, $(x+y)^p = x^p + y^p$.

(Hint. You are allowed to use binomial expansion without proof.

$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ where $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$, and $\binom{n}{i} \in \mathbb{Z}$.

Prove $p \mid \binom{p}{i}$ if $0<i<p$ and p is prime.

Deduce the claim.)

3. (a) Find a zero-divisor in $\mathbb{Z}_5[i] = \{a+bi \mid a,b \in \mathbb{Z}_5\}$

where $(a+bi)(c+di) = (ac-bd) + (ad+bc)\, i$.

(You do not need to show it is a ring.)

(b) Show that x^2+1 has no zero in $\mathbb{Z}_7$.

(c) Show that, if either $a \neq 0$ or $b \neq 0$ in $\mathbb{Z}_7$, then $a^2+b^2 \neq 0$ in $\mathbb{Z}_7$.

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(d) Show that $\mathbb{Z}_7[i]=\{a+bi \mid a,b\in\mathbb{Z}_7\}$ is a field.

(Hint. (c) if $a\neq 0$, then $a^2+b^2=a^2(1+(\frac{b}{a})^2)$; use (b).

(d) It is enough to show $\mathbb{Z}_7[i]$ is an integral domain. (why?)

$(a+bi)(c+di)=0 \Rightarrow (a+bi)(a-bi)(c-di)=0$

$\Rightarrow (a^2+b^2)(c^2+d^2)=0$ in $\mathbb{Z}_7$ use (c) to show either $a+bi=0$ or $c+di=0$.)

4. Show that the characteristic of an integral domain is either zero or prime.

5. Find the characteristic of $\mathbb{Z}_4\times\mathbb{Z}_6$ and $\mathbb{Z}_6\times\mathbb{Z}_8\times\mathbb{Z}_9$.