Lecture 14: Maximal ideals

Saturday, February 23, 2019 10:52 PM

<u>Recall</u>. Suppose A is a unital commutative ring. An ideal I of A is called <u>maximal</u> if ① I is proper; that means $I \neq A$; ② $\left.\begin{array}{l} I \subseteq J \\ J \triangleleft A \end{array}\right\} \Rightarrow$ either $J = I$ or $J = A$.

<u>Theorem</u>. Suppose A is a unital commutative ring and $I \triangleleft A$; then I is maximal $\Leftrightarrow A/I$ is a field.

<u>Pf.</u> ($\Rightarrow$) To show A/I is a field we have to show ① it is a unital commutative ring ② it is not the zero ring ③ any non-zero element is a unit.

① $(a_1+I)(a_2+I) = a_1a_2 + I = a_2a_1 + I = (a_2+I)(a_1+I)$.

$(1+I)(a+I) = (1\cdot a)+I = a+I$; similarly
$(a+I)(1+I) = a+I$ and so $1+I$ is the identity of A/I.

② $A/I \neq 0$ as $I \neq A$.

③ Suppose $a+I \neq 0+I$. Then $a \notin I$.

<u>Claim</u>. Smallest ideal J_0 of A that contains I as a subset and a as an element is $\{ar+x \mid r \in A, x \in I\}$.

Lecture 14: Maximal ideals

Saturday, February 23, 2019 11:08 PM

Pf of claim. First we show

$$J_0 = \{ar+x \mid r\in A, x\in I\} \text{ is an ideal.}$$

- $(ar_1+x_1)+(ar_2+x_2) = a\underbrace{(r_1+r_2)}_{\text{in } A} + \underbrace{(x_1+x_2)}_{\text{in } I} \in J_0.$

- $$a\underbrace{(r'r)}_{\text{in } A} + \underbrace{(r'x)}_{\substack{\text{in } I \\ \text{as } x\in I}} \in J_0.$$

2nd we observe. $\cdot\ a = (a)(1)+(0) \in J_0$

$\cdot\ \forall x\in I,\quad x = (a)(0)+x \in J_0 \Rightarrow I\subseteq J_0.$

3rd. If $J \triangleleft A$, $I\subseteq J$, and $a\in J$, then

$\forall x\in I, \forall r\in A,\quad ra\in J$ and $ra+x\in J$. Hence

$J_0 \subseteq J$. ; and claim follows. $\square$

Since I is a maximal ideal, $I\subseteq J_0$, and $a\in J_0\setminus I$,

we deduce that $J_0 = A$. Hence $1\in J_0$ which implies

$\exists r\in A, x\in I$ s.t. $ar+x=1$. Therefore

$1-ar = x\in I$; and so $1+I = ar+I = (a+I)(r+I)$.

This implies $a+I \in (A/I)^\times$.

Lecture 14: Maximal ideals

Saturday, February 23, 2019 11:19 PM

($\Leftarrow$) Since A/I is a field, $A/I \neq 0$; and so $I \neq A$.

Suppose to the contrary that I is not a maximal ideal.

Since $I \neq A$, we deduce that

$$\exists\ J \lhd A \text{ such that } I \subsetneq J \subsetneq A. \quad (*)$$

Suppose $a \in J \setminus I$. Hence $a + I \neq 0 + I$ in A/I. As A/I is a field, $\exists\ a' \in A$ s.t. $(a+I)(a'+I) = 1 + I$.

And so $\exists\ x \in I$ st. $1 - aa' = x$. Thus

$$1 = aa' + x \text{ for some } a' \in A \text{ and } x \in I.$$

$$\left.\begin{array}{l} a \in J \Rightarrow aa' \in J \\ x \in I \subseteq J \end{array}\right\} \Rightarrow aa' + x \in J \Rightarrow 1 \in J \Rightarrow J = A$$

which contradicts $(*)$. ∎

<u>Corollary</u>. Suppose A is a unital commutative ring. If I is a maximal ideal, then I is a prime ideal.

<u>Pf.</u> I : maximal $\Rightarrow A/I$: field

$\Rightarrow A/I$: integral domain

$\Rightarrow I$: prime. ∎

Lecture 14: Maximal ideals

Saturday, February 23, 2019 11:30 PM

<u>Corollary</u>. Suppose A is a unital commutative ring, $I \lhd A$, and $|A/I| < \infty$; then

$$I \text{ is maximal} \iff I \text{ is prime}.$$

<u>Pf.</u> I is maximal $\iff A/I$ is a field

because $|A/I| < \infty$

$\iff A/I$ is an integral domain

$\iff I$ is prime. ∎

Lecture 14: Examples

Friday, September 1, 2017 8:46 AM

Ex. Determine all the prime and maximal ideals of $\mathbb{Z}$.

Solution. Any ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$.

To determine, if $n\mathbb{Z}$ is either prime or maximal, we need to study the quotient ring $\mathbb{Z}/n\mathbb{Z}$.

We know that, if $n \geq 2$, then $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$. And $\mathbb{Z}_n$ is an integral domain $\Leftrightarrow$ $\mathbb{Z}_n$ is a field $\Leftrightarrow$ n is a prime.

- If $n=1$, then $n\mathbb{Z} = \mathbb{Z}$; and so it is neither prime nor maximal
- If $n=0$, then $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}$; which is an integral domain, but not a field. So $\{0\}$ is a prime ideal, but not a maximal ideal. Overall we have:

the set of maximal ideals of $\mathbb{Z} = \{p\mathbb{Z} \mid p \text{ is a prime number}\}$

the set of prime ideals of $\mathbb{Z} = \{n\mathbb{Z} \mid n \text{ is either } 0 \text{ or a prime number}\}$.

Next we show $m_\alpha(x)\mathbb{Q}[x]$ is a maximal ideal and deduce

Lecture 14: Maximal ideals, irreducible elements

Friday, September 1, 2017 8:26 AM

$\mathbb{Q}[\alpha] \simeq \mathbb{Q}[x]/m_\alpha(x)\mathbb{Q}[x]$ is a field. We have already proved that $m_\alpha(x)$ is irreducible in $\mathbb{Q}[x]$; so the following proposition gives us the above claim.

<u>Theorem</u>. Let R be a PID, and $a \in R \setminus \{0\}$.

Then $I = aR$ is maximal if and only if a is irreducible.

<u>Pf.</u> ($\Rightarrow$) We have to show
{assumption}
- $a \neq 0$, $a \notin R^\times$
- If $a = bc$, then either $b \in R^\times$ or $c \in R^\times$.

- Since I is maximal, it is a proper ideal. So $a \notin R^\times$.

$$\left.\begin{array}{l} \cdot\ a = bc \in aR \\ aR \text{ maximal} \Rightarrow aR \text{ prime} \end{array}\right\} \Rightarrow \text{either } b \in aR \text{ or } c \in aR.$$

If $b \in aR$, then $b = ab'$. Hence $a = bc = ab'c$; and so by the cancellation law $b'c = 1$ which implies $c \in R^\times$.

Similarly, if $c \in bR$, we can deduce that $b \in R^\times$.

($\Leftarrow$) Suppose a is irreducible. Then $a \notin R^\times$; and so $1 \notin aR$, which implies $aR \neq R$. Now suppose $aR \subseteq J$ and $J \triangleleft R$.

Since R is a PID, $J = bR$ for some b. As

$a \in aR \subseteq bR$, $\exists r \in R$ such that $a = br$. As a is

irreducible, either b is a unit or r is a unit.

If $b \in R^\times$, $J = bR = R$. If r is a unit,

$b = ar^{-1} \in aR$; and so $bR \subseteq aR$. On the other hand,

$aR \subseteq bR$; therefore $J = bR = aR$.

So aR is maximal. ∎

Corollary Let D be a PID. Suppose a is irreducible in D.

Then $D/\langle a \rangle$ is a field.

Corollary. If $\alpha \in \mathbb{C}$ is an algebraic number, then $\mathbb{Q}[\alpha]$ is a field.

Pf. Earlier based on the 1st isomorphism theorem we proved $\mathbb{Q}[\alpha] = \operatorname{Im} \phi_\alpha \simeq \mathbb{Q}[x]/\ker \phi_\alpha = \mathbb{Q}[x]/m_\alpha(x)\mathbb{Q}[x]$.

We have also proved that $m_\alpha(x)$ is irreducible in $\mathbb{Q}[x]$.

Lecture 14: Q[an algebraic number] is a field

Monday, August 28, 2017 12:37 AM

Hence by the previous corollary and the fact that $\mathbb{Q}[x]$ is a PID, we get the claim. $\blacksquare$