Maximal and prime ideals

Sunday, August 27, 2017 9:21 PM

Instead of directly proving the third part of the theorem, we will investigate the following questions:

Suppose R is a unital commutative ring. Under what conditions on an ideal I of R do we have that R/I is a field?

Under what conditions on I do we have that R/I is an integral domain?

Investigation.

Since R is a unital commutative ring, so is R/I. So

R/I is a field $\Leftrightarrow$ (1) R/I is not the zero ring; that means it has at least two elements.

(2) $U(R/I) = (R/I) \setminus \{0+I\}$.

$\Leftrightarrow$ (1) $R \neq I$.

(2) $x+I \neq 0+I$ implies $\exists r \in R$ such that $rx+I = 1+I$.

$\Leftrightarrow$ (1) $R \neq I$

(2) $x \notin I$ implies $\exists y \in I, r \in R, rx+y=1$.

Maximal ideals

Sunday, August 27, 2017 9:48 PM

$\Rightarrow$ ① $I \neq R$

② $\forall x \in R\setminus I$, if $J \lhd R$, $x \in J$, $I \subseteq J$, then $1 \in J$

$\Rightarrow$ ① $I \neq R$

② $(J \lhd R$ and $I \subsetneq J) \Rightarrow J = R$.

Def. We say I is a maximal ideal of R if

① I is a proper ideal; this means $I \neq R$

② If $J \lhd R$ and $I \subsetneq J$, then $J = R$.

Theorem. Let R be a unital commutative ring. Then I is a maximal ideal of R if and only if R/I is a field.

Pf. $(\Leftarrow)$ we have already proved.

$(\Rightarrow)$ Since I is a proper ideal, R/I is a non-zero ring.

$\forall x+I \in (R/I)\setminus\{0+I\} \Rightarrow x \in R\setminus I$

$\Rightarrow$ the ideal generated by $\{x\}\cup I$ is R as I is a maximal ideal.

Claim. $\langle \{x\}\cup I\rangle = \{rx+y \mid r\in R, y \in I\}$.

Pf of claim. $x \in \{x\}\cup I \Rightarrow \forall r \in R, y \in I$, $rx+y \in \langle\{x\}\cup I\rangle$.

Maximal ideals

Sunday, August 27, 2017 10:02 PM

And so $\underbrace{\{rx+y \mid r\in R, y\in I\}}_{J} \subseteq \langle \{x\}\cup I\rangle$. Next we will show that J is an ideal of R:

- $(\underbrace{r_1}_{\text{in } R} x+\underbrace{y_1}_{\text{in } I})+(\underbrace{r_2}_{\text{in } R} x+\underbrace{y_2}_{\text{in } I}) = (\underbrace{r_1+r_2}_{\text{in } R})\, x + (\underbrace{y_1+y_2}_{\text{in } I}) \in J$

- $\forall r, r'\in R, y\in I,\quad r'(rx+y) = (\underbrace{r'r}_{\text{in } R})\, x + (\underbrace{r'y}_{\text{in } I}) \in J$

Since R is unital and $0\in I$, $x\in J$. And since $(0)(x)=0$, $I\subseteq J$. So $\{x\}\cup I\subseteq J$, which implies $\langle\{x\}\cup I\rangle\subseteq J$. And the claim follows.

. Since $\langle\{x\}\cup I\rangle = R$, we deduce $\exists\, r\in R, y\in I$ s.t. $rx+y=1$. Hence $rx+I = 1+I$; and so

$$(r+I)(x+I) = 1+I.$$

As R/I is commutative, we get that $x+I\in U(R/I)$. Therefore R/I is a field. ■