Wednesday, May 4, 2016 11:06 AM

1. (a) Prove that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n\,\varphi(n)}$ is convergent, where $\varphi(n)$ is the Euler-Phi function.

(b) $\sum_{n\le x} \frac{1}{\varphi(n)} = C \ln x + O(1)$ for some constant $C$.

[Hint. In your exam, you have proved $\frac{1}{\varphi(n)} = \frac{1}{n} \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}$.

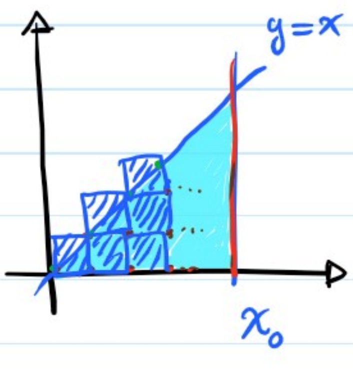
So
$$\sum_{n\le x} \frac{1}{\varphi(n)} = \sum_{n\le x} \frac{1}{n} \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$
$$\stackrel{?}{=} \sum_{\substack{d,d' \\ dd'\le x}} \frac{1}{dd'} \cdot \frac{\mu(d)^2}{\varphi(d)}$$
$$= \sum_{d\le x} \frac{\mu(d)^2}{d\cdot\varphi(d)} \Big(\sum_{d'\le \frac{x}{d}} \frac{1}{d'}\Big)$$

Use $\sum_{m\le y} \frac{1}{m} = \ln y + O(1)$. ]

2. Let $\sigma = \mathbb{1} * \mathrm{id}$. Prove that
$$\frac{1}{n} \sum_{m=1}^{n} \sigma(m) = \frac{\pi^2}{12} n + O(\ln n).$$

[In average the sum of positive divisors of integers $\le n$ is growing like $\frac{\pi^2}{12} n$. (notice that $\frac{\pi^2}{12} < 1$.)]

[Hint:
$$\sum_{m=1}^{n} \sigma(m) = \sum_{m=1}^{n} \sum_{d|m} d = \sum_{d.d'\le n} d$$
$$= \sum_{d'=1}^{n} \sum_{d\le n/d'} d$$




$\sum_{d\le x_0} d$ is the number of integer points in the triangle (including on the line $x=x_0$)

To each point attach ▨ a unit square

$\sum$

So $\sum_{d \leq x_0} d$ = area of triangle $+ O(x_0)$

$$= \frac{1}{2} x_0^2 + O(x_0).$$

And $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. ]

3. Suppose $a$ is a positive, odd integer, which is not a perfect square.

(a) Prove that there are infinitely many primes $p$ such that

$a$ is a residue mod $p$.

(b) Prove that there are infinitely many primes $p$ such that

$a$ is a non-residue mod $p$.

[Remark. (1) The assumptions "$a$ is positive and odd" are not needed.

(2) The second part implies that, if $x^2 - a = 0$ has a solution mod $p$ for all large enough $p$, then $a$ is a perfect square.]

[Hint. Use quadratic reciprocity, Chinese remainder theorem, and Dirichlet theorem.]