Conditional, proofs.

Wednesday, September 30, 2015 10:57 AM

In the previous lecture we defined <u>conditional propositions</u> a.k.a. <u>implications</u>.

P implies Q. $P \Rightarrow Q$.

If P, then Q. And its truth table is

P is sufficient for Q.

Q is necessary for P.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Since in mathematics we often deal with this type of propositions, let's try to find new forms of such propositional form.

What does it mean for $P \Rightarrow Q$ to fail? For you to show me this implication fails, you have to provide a situation where P is true and Q is false; which means

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q).$$

Let's double check this using the truth table.

P	Q	$\neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$P \wedge (\neg Q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

So by de Morgan's law we have

$$
\begin{aligned}
P \Rightarrow Q &\equiv \neg(P \wedge \neg Q) \equiv \neg P \vee Q \\
&\equiv Q \vee (\neg P) \\
&\equiv (\neg Q) \Rightarrow (\neg P).
\end{aligned}
$$

- $\neg Q \Rightarrow \neg P$ is called the <u>contrapositive</u> of $P \Rightarrow Q$, and it is a useful method to prove things.
- Before we see some examples, let me warn you that

$$P \Rightarrow Q \not\equiv \underbrace{Q \Rightarrow P}$$

is called the <u>converse</u> of $P \Rightarrow Q$.

<u>Ex.</u> (Kenken)

it is unique

In the blue box we can have either $\boxed{\begin{matrix}1\\3\end{matrix}}$ or $\boxed{\begin{matrix}3\\1\end{matrix}}$.

If the second case happens, we would have two 3s in a row, which is a contradiction. So the first case happens.

a row, which is a contradiction. So the first case happens.

In the yellow box there are two possible cases $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ or $\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$.

If the first case happens, we would get two 1s in a row, which is a contradiction. Hence the second case is true.

The only remaining possibility for is 3.

Similarly we have that is 2.

Using the same logic we have that and are 2 and 1, respectively.

In this game, you see how we use case-by-case proof together with proof by contradiction together in our daily games or decisions.

<u>Def.</u> Suppose m and n are two integers. We say m divides n if for some integer k we have

$$n = mk.$$

(We also say m is a divisor of n, or n is a multiple of m) We denote it by $m | n$.

n is a multiple of m) We denote it by $m \mid n$.

<u>Ex.</u> $1 \mid n$ for any integer n.

<u>Pf</u>. For any integer n, $n = (n)(1)$. So n is a multiple of 1. ■

<u>Ex</u>. For non-zero integers a and b, $a \mid b \Rightarrow |a| \leq |b|$.

<u>Pf.</u> $a \mid b \Rightarrow$ for some integer k, $b = ak$

$$\Rightarrow |b| = |a||k|.$$

<u>Claim</u> $k \neq 0$.

<u>Pf of claim</u>. Suppose to the contrary that $k = 0$. Then $b = (a)(0) = 0$, which contradicts the assumption that b is non-zero.

Since k is a non-zero integer, we have $|k| \geq 1$.

Hence $|b| = |a||k| \geq |a|$ as $|a| \geq 0$. ■

<u>Warning</u>. By <u>multiple</u>, we mean integer multiple. We are NOT allowed to multiply by fractions.

[We also discussed that, if $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then P and Q are equivalent. And we showed this using the truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T

both are true.

T	T	T	T	T	both are true.
T	F	F	T	F	
F	T	T	F	F	
F	F	T	T	T	both are false.]