Proofs: inequality and divisibility.

Wednesday, October 7, 2015 11:02 AM

We ended the previous lecture with the following statement:

<u>Ex</u>. For any real numbers x and y,

$$x^2 \le y^2 \iff |x| \le |y|.$$

<u>Pf</u>. <u>Step 1</u>. For any real number x, $x^2 = |x|^2$.

<u>Pf of step 1</u>. <u>Case 1</u>. $x \ge 0 \Rightarrow |x| = x \Rightarrow |x|^2 = x^2$.

(Case-by-case)

<u>Case 2</u>. $x < 0 \Rightarrow |x| = -x \Rightarrow |x|^2 = (-x)^2 = x^2$.

<u>Step 2</u>. ($\Rightarrow$) Suppose to the contrary $|x| \not\le |y|$.

So $|x| > |y| \Rightarrow \left\{ \begin{array}{l} |x||x| > |y||x| \Rightarrow |x|^2 > |xy| \\ |x||y| > |y||y| \Rightarrow |xy| > |y|^2 \end{array} \right\} \Rightarrow$

$|x|^2 > |y|^2 \Rightarrow x^2 > y^2$ which contradicts the assumption that $x^2 \le y^2$.

<u>Step 3</u>. ($\Leftarrow$) $|x| \le |y|$ then the same argument as above shows $x^2 \le y^2$. ∎

<u>Ex</u>. For any real numbers x and y,

$$x^2 + y^2 \ge 2|xy|.$$

<u>Pf</u>. Let's try to reconstruct the inequality by going "backward".

Warning this does NOT always work.

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$$x^2+y^2 \geq 2|xy| \Longleftarrow x^2+y^2-2|xy| \geq 0$$

$$\Leftarrow |x|^2+|y|^2-2|x||y| \geq 0$$ (Step 1 in the prev. ex.)

$$\Leftarrow \underbrace{(|x|-|y|)^2 \geq 0}$$

Hooray! this is true!

So now you can write the proof:

For any real number z, $z^2 \geq 0 \Rightarrow (|x|-|y|)^2 \geq 0$

$$\Rightarrow |x|^2+|y|^2-2|x||y| \geq 0$$

$$\Rightarrow x^2+y^2 \geq 2|xy|$$

as $|x|^2=x^2$. ∎

You might need a "bag of tools" to prove an equality.

Ex. For any real numbers x, y, z,

$$x^2+y^2+z^2 \geq xy+xz+yz.$$

Pf. Solution 1. Let's view the whole thing as a function of x.

$$x^2-(y+z)x+(y^2+z^2-yz) \geq 0 \quad \Leftarrow$$

$$x^2-(y+z)x+\frac{(y+z)^2}{4}-\left(\frac{y^2+2yz+z^2}{4}\right)+(y^2+z^2-yz) \geq 0 \Leftarrow$$

$$\left(x-\frac{y+z}{2}\right)^2+\frac{3y^2+3z^2-6yz}{4} \geq 0 \quad \Leftarrow$$

$$\left(x-\frac{y+z}{2}\right)^2 + \frac{3}{4}(y^2+z^2-2yz) \geq$$

$$\left(x-\frac{y+z}{2}\right)^2+\frac{3}{4}\left(y^2+z^2-2yz\right)\geq 0 \quad \Leftarrow$$

$$\left(x-\frac{y+z}{2}\right)^2+\frac{3}{4}(y-z)^2\geq 0$$

Hooray! it is true as $\left(x-\frac{y+z}{2}\right)^2\geq 0$, $\frac{3}{4}>0$, and $(y-z)^2\geq 0$.

<u>Solution 2.</u> By the previous example:

$$\left.\begin{aligned} x^2+y^2 &\geq 2xy \\ x^2+z^2 &\geq 2xz \\ y^2+z^2 &\geq 2yz \end{aligned}\right\} \xrightarrow{\text{(by adding)}} 2(x^2+y^2+z^2)\geq 2(xy+xz+yz)$$

$$\Rightarrow x^2+y^2+z^2\geq xy+xz+yz.$$

$\blacksquare$

<u>Ex</u>. Prove that the function $f(x)=\sqrt{x+2}$, where $x\geq -2$, is increasing.

<u>Pf</u>. We need to prove for any $x, y\geq -2$,

$$x\leq y \Rightarrow f(x)\leq f(y).$$

i.e. $$x\leq y \Rightarrow \sqrt{x+2}\leq\sqrt{y+2}.$$

Suppose to the contrary that $\sqrt{x+2}\nleq\sqrt{y+2}$.

So $\sqrt{x+2}>\sqrt{y+2}$. Therefore by the first example

$\left(\sqrt{x+2}\right)^2>\left(\sqrt{y+2}\right)^2$ (notice that $\sqrt{x+2}$, $\sqrt{y+2}$ are non-negative.)

$\Rightarrow x+2 > y+2 \Rightarrow x > y$ which contradicts the assumption that $x \leq y$. $\blacksquare$

Ex. For integers x and y,

$$xy \text{ is odd} \iff x \text{ and } y \text{ are odd}.$$

(look at my notes for the previous lecture.)