Strong induction

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Ex. Suppose $a_0 = 0$, $a_1 = 3$, and $a_{n+1} = a_n + 2a_{n-1}$

Find a_{100}.

Solution. Let's start with inductive reasoning to guess what a_n should be. [I am too lazy to compute a_{100} by hand.]

$$0,\ 3,\ 3,\ 9,\ 15,\ 33,\ 63,\ 129,\ \ldots$$

Any guess? If not, let's do one more 255.

Are these numbers similar to the ones in 2048 game!?

$$2^0-1 \qquad 2^2-1 \qquad 2^4-1 \qquad 2^6-1$$

$$2^1+1 \qquad 2^3+1 \qquad 2^5+1 \qquad 2^7+1$$

Conjecture $a_n = \begin{cases} 2^n - 1 & \text{if } n \text{ is even} \\ 2^n + 1 & \text{if } n \text{ is odd} \end{cases}$

I.e. $a_n = 2^n - (-1)^n$.

If this conjecture has an affirmative answer, we get

$a_{100} = 2^{100} - 1$.

How can we prove this conjecture?

Since a_n is defined recursively, it is a good idea to try induction.

$$a_{k+1} = a_k + 2a_{k-1} \quad \text{unless} \quad k=0 \text{ or } 1.$$

$$= (2^k - (-1)^k) + 2\,(2^{k-1} - (-1)^{k-1})$$

$$= 2^k - (-1)^k + 2^k - 2\,(-1)^{k-1}$$

$$= (2^k + 2^k) - [-(-1)^{k+1} + 2\,(-1)^{k-1}]$$

$$= (2)(2^k) - (-1)^{k+1}$$

$$= 2^{k+1} - (-1)^{k+1} \quad \text{as we wished.}$$

This is NOT the induction that we used before. Here we need to go back one step and two steps. Sometimes we need to go back even further. This is called strong induction.

Strong induction.

Base of strong induction. "$P(n_0)$ is true."

Inductive step. For any integer $k \geq n_0$

$P(i)$ for $1 \leq i \leq k \Rightarrow P(k+1)$. "is true"

Then For any integer $n \geq n_0$, "$P(n)$ is true".

Ex. Any integer $n \geq 2$ can be written as product of primes.

primes.

<u>Definition</u>. An integer $n \geq 2$ is called prime if the only positive divisors of n are 1 and n.

<u>Recall</u>
- For any non-zero integers a and b, $a \mid b \Rightarrow |a| \leq |b|$.
- For any positive numbers x and y, $\min\{x,y\} \leq \sqrt{xy}$.

If n is NOT prime, then it has a positive divisor d other than 1 and n. So

$$\left.\begin{array}{l} 1 \leq d \leq n \\ d \neq 1 \\ d \neq n \end{array}\right\} \Rightarrow 1 < d < n$$

and $n = dk$ for some integers k.

$$\left.\begin{array}{l} 0 < n = dk \\ 0 < d \end{array}\right\} \Rightarrow 0 < k.$$

$$\left.\begin{array}{l} 1 < d \\ 0 < k \end{array}\right\} \Rightarrow k < dk = n, \qquad \left.\begin{array}{l} d < n = dk \\ 0 < d \end{array}\right\} \Rightarrow 1 < k.$$

Altogether $n = dk$ for some integers $1 < d, k < n$.

And we get the following lemma.

<u>Lemma</u>. For any integer $n \geq 2$, if n is NOT prime, then there are integers n_1 and n_2 such that

- $2 \leq n_1, n_2 \leq n-1$

- $2 \leq n_1, n_2 \leq n-1$
- $n = n_1 \cdot n_2$

Remark. In fact we have $\min\{n_1, n_2\} \leq \sqrt{n}$. So to check if a positive integer is prime or not, it is enough to look at integers $1 \leq m \leq \sqrt{n}$.

Proof of Example. We use strong induction.

Base of induction. $n=2$ is prime. So there is nothing to prove.

(I am using this convention that having a single term is still considered to be a "product", e.g. $\prod_{i=1}^{n} a_i$ is the product of a_i's even if $n=1$.)

Strong inductive step. We need to show: for any integer $k \geq 2$

Any integer $2 \leq i \leq k$ can be written as product of primes (Strong induction hypothesis.)

$$\Downarrow$$

$k+1$ can be written as product of primes.

Case 1. If $k+1$ is prime, then it is written as product of primes (by our convention).

Case 2. If $k+1$ is NOT prime, then, by Lemma, there are integers k_1 and k_2 such that

- $2 \leq k_1, k_2 \leq (k+1)-1 = k$

- $2 \leq k_1, k_2 \leq (k+1)-1 = k$
- $k+1 = k_1 \cdot k_2$.

By the strong induction hypothesis, k_1 and k_2 can be written as product of primes. So $k_1 \cdot k_2$ can be written as product of primes. And we are done as $k+1 = k_1 k_2$. $\blacksquare$