Friday, October 16, 2015 11:03 AM

Let $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for any positive integer n. This is called the Fibonacci sequence.

<u>Theorem</u>. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then for any positive integer n,

$$A^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}.$$

<u>Proof</u>. We use induction on n.

<u>Base of induction</u>. <u>$n=1$</u>.

$$A^1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_0 & f_1 \\ f_1 & f_2 \end{bmatrix}. \checkmark$$

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 0 + 1 = 1.$$

<u>Inductive step</u>. For any positive integer k,

$$A^k = \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix} \overset{?}{\Longrightarrow} A^{k+1} = \begin{bmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{bmatrix}$$

$$A^{k+1} = A^k \cdot A$$

$$= \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$ (induction hypothesis)

$$= \begin{bmatrix} f_k & f_{k-1} + f_k \\ f_{k+1} & f_k + f_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} f_k & f_{k+1} \\ f_k & f_k \end{bmatrix}$$ (definition of f_m).

$$= \begin{bmatrix} \cdots & \cdots \\ f_{k+1} & f_{k+2} \end{bmatrix} \quad \blacksquare$$

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We get several important properties of the Fibonacci sequence using this.

<u>Corollary</u> For any positive integer n,

$$f_{n-1} \cdot f_{n+1} - f_n^2 = (-1)^n.$$

<u>Proof.</u> $A^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix} \Rightarrow \det(A^n) = \det \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$

$$\Rightarrow f_{n-1} \cdot f_{n+1} - f_n^2 = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right)^n = (-1)^n. \quad \blacksquare$$

<u>Corollary</u>. For any positive integers m and n,

$$f_{n+m} = f_{n+1} f_m + f_n f_{m-1}.$$

<u>Proof.</u> $A^{m+n} = A^m \cdot A^n$

$$\Rightarrow \begin{bmatrix} f_{m+n-1} & f_{m+n} \\ f_{m+n} & f_{m+n+1} \end{bmatrix} = \begin{bmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{bmatrix} \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} f_{m-1} f_{n-1} + f_m f_n & f_{m-1} f_n + f_m f_{n+1} \\ f_m f_{n-1} + f_{m+1} f_n & f_m f_n + f_{m+1} f_{n+1} \end{bmatrix}$$

Compare the <u>1,2</u> entry:

$$f_{m+n} = f_{m-1} f_n + f_m f_{n+1}. \quad \blacksquare$$

Theorem. For any positive integers m and n,

$$m \mid n \iff f_m \mid f_n .$$

Proof. I prove only ($\Rightarrow$). ($\Leftarrow$) will be one of your exercises.

- $m \mid n \Rightarrow n = mk$ for some integer k.

  Since m and n are positive, so is k.

- So we have to show: for any positive integers m and k

$$f_m \mid f_{mk} .$$

We use induction on k.

Base. $k=1$.

It is clear as $f_m = 1 \times f_m$.

Inductive step. For any positive integer k',

for any positive integer m, $f_m \mid f_{mk'} \overset{?}{\Longrightarrow}$ for any positive integer m, $f_m \mid f_{m(k'+1)}$

$$\begin{aligned} f_{m(k'+1)} &= f_{mk'+m} \qquad \text{I} \\ &= f_{m-1} f_{mk'} + f_m f_{mk'-1} \end{aligned}$$

(by the previous corollary.)

$f_m \mid f_{mk'} \Longrightarrow f_{mk'} = f_m \cdot r$ for some integer r. II

(by induction hypothesis)

I, II $\Longrightarrow f_{m(k'+1)} = f_{m-1} \cdot f_m \cdot r + f_m \cdot f_{mk'-1}$

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$$= f_m \left(\underbrace{f_{m-1} + f_{mk'-1}}_{\text{integer}} \right)$$

$$\Longrightarrow \quad f_m \mid f_{m(k'+1)} . \qquad \blacksquare$$