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In class you have asked how you should deal with a bit more general sequences than the Fibonacci sequence. In this note, you can learn a bit more about these sequences.

Let $a_0 = a, a_1 = b$ and for any positive integer n,

$$a_{n+1} = x\, a_n + y\, a_{n-1}.$$

Let $A = \begin{bmatrix} 0 & 1 \\ y & x \end{bmatrix}$.

<u>Lemma</u>. For any positive integer n,

$$A \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

<u>Proof</u>. $$A \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ y & x \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_n \\ y\, a_{n-1} + x\, a_n \end{bmatrix}$$
$$= \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}. \qquad \blacksquare$$

<u>Corollary</u> For any non-negative integer n and m

$$A^n \begin{bmatrix} a_m \\ a_{m+1} \end{bmatrix} = \begin{bmatrix} a_{n+m} \\ a_{n+m+1} \end{bmatrix}$$

<u>Proof.</u> We use induction on $\underline{n}$.

<u>Base of induction</u> $\underline{n=0}$.

$$a \quad \big] = \begin{bmatrix} a_m \\ \end{bmatrix}$$

$k = 1 \Rightarrow \text{LHS} = \begin{bmatrix} \cdots \\ a_{m+1} \end{bmatrix}$. $\text{RHS} = \begin{bmatrix} \cdots \\ a_{0+m+1} \end{bmatrix} = \begin{bmatrix} \cdots \\ a_{m+1} \end{bmatrix} = \text{LHS}$.

Inductive step. For any non-negative integer k,

For any non-negative integer m, $\overset{?}{\Longrightarrow}$ For any non-neg. int m,

$$A^k \begin{bmatrix} a_m \\ a_{m+1} \end{bmatrix} = \begin{bmatrix} a_{m+k} \\ a_{m+k+1} \end{bmatrix} \qquad\qquad A^{k+1} \begin{bmatrix} a_m \\ a_{m+1} \end{bmatrix} = \begin{bmatrix} a_{m+k+1} \\ a_{m+k+2} \end{bmatrix}.$$

$$A^{k+1} \begin{bmatrix} a_m \\ a_{m+1} \end{bmatrix} = A \cdot A^k \begin{bmatrix} a_m \\ a_{m+1} \end{bmatrix} = A \begin{bmatrix} a_{m+k} \\ a_{m+k+1} \end{bmatrix} \quad \text{(by induction hypothesis)}$$

$$= \begin{bmatrix} a_{m+k+1} \\ a_{m+k+2} \end{bmatrix} \quad \text{(by Lemma)}. \blacksquare$$

Theorem. For any positive integer n,

$$A^n \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{bmatrix}.$$

Proof. By the previous corollary, $A^n \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$ and $A^n \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix}$. And one gets the above equation multiplying A^n by the columns of $\begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$. $\blacksquare$

Corollary. For any non-negative integers n,

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n (a_0 a_2 - a_1^2).$$

Proof. By the above theorem,

$$A^n \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{bmatrix}.$$

$8 \quad \left| \left(\begin{bmatrix} 0 & 1 \\ \cdots \end{bmatrix} \right)^n \right| \left| \begin{pmatrix} a_0 & a_1 \\ \cdots \end{pmatrix} \right| \quad \left| \begin{pmatrix} a_n & a_{n+1} \\ \cdots \end{pmatrix} \right|$

So $\quad \det\left(\begin{bmatrix}0 & 1\\ y & x\end{bmatrix}\right)^n \det\begin{pmatrix}a_0 & a_1\\ a_1 & a_2\end{pmatrix} = \det\begin{pmatrix}a_n & a_{n+1}\\ a_{n+1} & a_{n+2}\end{pmatrix}$

$\Rightarrow \quad (-y)^n\,(a_0a_2-a_1^2) = a_n\,a_{n+2}-a_{n+1}^2 .$ ∎

One way to find a_n is by finding eigenvalues of A. If they are distinct, then A is similar to a diagonal matrix:

$$A = S\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}S^{-1}.$$

So $A^n = S\begin{bmatrix}\lambda_1^n & \\ & \lambda_2^n\end{bmatrix}S^{-1}$. Then using the above theorem we can find $\underline{a_n}$.

Recall that eigenvalues of $\begin{bmatrix}0 & 1\\ y & x\end{bmatrix}$ are roots of

$$\lambda^2 - x\lambda - y = 0.$$

Summary. If $\lambda^2 - x\lambda - y = 0$ has distinct solutions λ_1 and λ_2, then $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ for some numbers c_1 and c_2. You can find them using

$$\begin{cases} a_0 = c_1 + c_2\\ a_1 = \lambda_1 c_1 + \lambda_2 c_2 .\end{cases}$$