Friday, October 23, 2015 10:58 AM

Proposition $A \triangle B = (A \cup B) \setminus (A \cap B)$

$= (A \setminus B) \cup (B \setminus A)$.

Proof. We use a truth-table to prove this:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in A \cap B$	$x \in A \setminus B$	$x \in B \setminus A$	$x \in (A \cup B) \setminus (A \cap B)$	$x \in (A \setminus B) \cup (B \setminus A)$	$x \in A \triangle B$
T	T	T	T	F	F	F	F	F
T	F	T	F	T	F	T	T	T
F	T	T	F	F	T	T	T	T
F	F	F	F	F	F	F	F	F

They have the same truth-values which implies that

$$x \in (A \cup B) \setminus (A \cap B) \iff x \in (A \setminus B) \cup (B \setminus A)$$

$$\iff x \in A \triangle B.$$

Hence we have $A \triangle B = (A \cup B) \setminus (A \cap B)$

$= (A \setminus B) \cup (B \setminus A)$. ■

As in the case of propositional forms, there are very useful set equalities:

Proposition. For three sets $A, B, C \in P(X)$,

① $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

①' $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

② $(A \cup B)^c = A^c \cap B^c$.

②' $(A \cap B)^c = A^c \cup B^c$.

③ $A\setminus B = A\cap B$.

④ $A \subseteq B \iff A\cap B = A \iff A\cup B = B \iff A\setminus B = \emptyset$

⑤ $A\cap B \subseteq A \subseteq A\cup B$.

⑥ $A\cap B = \emptyset \iff A \subseteq B^c$.

Proof of which one of ②-⑥ do you want to see?

③ and parts of ④ were proved look at the bottom of this note for their proofs.

Quantifiers

Almost all the mathematical statements have quantifiers.

For instance we have seen:

Ex. For any integers m and n, mn is odd if and only if m and n are odd.

We use $\forall$ to say "for all" or "for any". So the above statement can be written as

$$\forall m,n \in \mathbb{Z}, \quad 2\nmid mn \iff 2\nmid m \vee 2\nmid n.$$

Here are other quantifiers:

$\exists x\in A, \cdots$: there exists x in A s.t. $\cdots$

or For some x in A, $\cdots$

$\nexists x\in A, \cdots$: there is NO x in A s.t. $\cdots$

Ex. Use quantifiers to say that minimum of A exists, where $A\subseteq\mathbb{R}$.

Solution. $\exists x\in A, \forall y\in A, \ x\le y$.

Ex. What happens if we switch the order of the quantifiers?

$\forall y \in A, \exists x \in A, x \leq y$

This is always true!

Ex. Use quantifiers to say $A \subseteq \mathbb{R}$ is bounded.

Solution. $\exists\, m, M \in \mathbb{R}, \forall x \in A, \quad m \leq x \leq M.$ ∎

Ex. Prove or disprove that any bounded subset of $\mathbb{R}$ has a minimum.

Solution. It is NOT true. To disprove it, we need to find a bounded set with **no** minimum.

Claim $(0,1)$ is bounded.

Proof of claim. $\forall x \in (0,1), \quad 0 \leq x \leq 1$. So it is bounded.

($m=0$ and $M=1$ satisfy the above conditions.)

Claim $(0,1)$ has no minimum.

Proof of claim. Suppose to the contrary that $x = \min(0,1)$.

So $x \in (0,1)$ I and $\forall y \in (0,1), \quad x \leq y$. II

I $\Rightarrow 0 < x \implies 0 < \frac{x}{2} \implies \frac{x}{2} < \frac{x}{2} + \frac{x}{2} = x$

So $\quad 0 < \frac{x}{2} < x < 1 \implies \frac{x}{2} \in (0,1) \wedge \frac{x}{2} \leq x$

Well-ordering principle $\forall\, \emptyset \neq A \subseteq \mathbb{Z}^+, \exists x \in A, \forall y \in A, x \leq y.$

(Any non-empty subset of positive integers has a minimum.)

$A \setminus B = A \cap B^c$

$x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B$

$\Leftrightarrow x \in A \wedge (x \in X \wedge x \notin B)$

$\Leftrightarrow x \in A \wedge x \in B^c$

$\Leftrightarrow x \in A \cap B^c.$

$A \subseteq B \Leftrightarrow A \setminus B = \emptyset$

($\Rightarrow$) Suppose to the contrary, $\exists x \in A \setminus B$. Then $x \in A \wedge x \notin B$.

$x \in A \Rightarrow x \in B$ since $A \subseteq B$.

This contradicts the fact that $x \notin B$.

($\Leftarrow$) Since $A \setminus B = \emptyset$, $x \notin A \setminus B$. Therefore

$\neg(x \in A \wedge x \notin B) \equiv x \notin A \vee x \in B.$

$\equiv (x \in A \Rightarrow x \in B)$

So $A \subseteq B$. ∎

$A \subseteq B \Leftrightarrow A \cup B = B$

(it is added to help you for the HW assignment.)

($\Rightarrow$) $x \in A \cup B \Rightarrow x \in A \vee x \in B$

<u>Case 1.</u> $x \in A \Rightarrow x \in B$ as $A \subseteq B$.

<u>Case 2.</u> $x \in B$.

So in either case we have

$x \in B \Rightarrow x \in A \vee x \in B \Rightarrow x \in A \cup B$

$(\Leftarrow)$ $x \in A \Rightarrow x \in A \vee x \in B \Rightarrow x \in A \cup B \underset{\boxed{A \cup B = B}}{\Rightarrow} x \in B$

So $A \subseteq B$. $\blacksquare$