Wednesday, November 4, 2015 11:05 AM

On Friday, we wanted to prove the following:

Theorem. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does NOT exist.

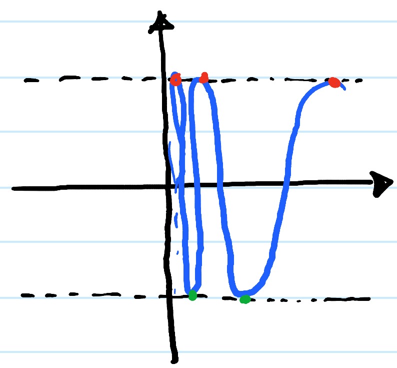
Proof. $\forall L,\ \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \neq L \iff$

(∀L, the first player has a "winning move": $\varepsilon_0>0$)

$\forall L, \exists \varepsilon_0>0,$ (that no matter what "my move" is, I lose.)

$$\forall L, \exists \varepsilon_0>0, \forall \delta>0, \exists x,\ 0<|x-0|<\delta \ \wedge\ \left|\sin\left(\tfrac{1}{x}\right)-L\right|>\varepsilon_0.$$

Claim $\varepsilon_0=\frac{1}{2}$ works.



$$\forall L, \forall \delta>0, \exists x,\ 0<|x|<\delta \ \wedge\ \left|\sin\left(\tfrac{1}{x}\right)-L\right|>\tfrac{1}{2}$$

- It is enough to find arbitrarily small numbers x and x' such that $\sin\left(\frac{1}{x}\right)=1$ and $\sin\left(\frac{1}{x'}\right)=-1$.

- **Why?** Because $\forall L$ either $|1-L|>\frac{1}{2}$ or $|-1-L|>\frac{1}{2}$.

- Notice that $\sin\left(2k\pi+\frac{\pi}{2}\right)=1$ and $\sin\left(2k\pi-\frac{\pi}{2}\right)=-1$ for any $k\in\mathbb{Z}$.

On the other hand, if $2k-\frac{1}{2}>\frac{1}{\pi\delta}$, then

$$\frac{1}{2k\pi + \frac{\pi}{2}} < \delta \quad \text{and} \quad \frac{1}{2k\pi - \pi/2} < \delta .$$

So overall we have

$\forall L, \forall \delta,$ either $x = \dfrac{1}{2k\pi - \pi/2}$ or $x = \dfrac{1}{2k\pi + \pi/2}$

is a "good move" if $k > \left(\frac{1}{\pi\delta} + 1/2\right)/2$.

as

$$0 < \left|\frac{1}{2k\pi - \pi/2}\right| < \delta \;\wedge\; 0 < \left|\frac{1}{2k\pi + \pi/2}\right| < \delta$$

$$\wedge \quad \sin\left(2k\pi - \pi/2\right) = -1 \;\wedge\; \sin\left(2k\pi + \frac{\pi}{2}\right) = 1 .$$

and at least one of them is away from L by at least $1/2$. $\blacksquare$

Another quanifier that we often use is $\exists!\, a \in A, \ldots$

there is a unique $a \in A$ such that ...

Ex. $\forall n \in \mathbb{Z}, \exists!\, m \in \{n, n+1\}, 2 \mid m$.

Remark. This is what you have proved in one of your HW assignments:

$\forall n \in \mathbb{Z}$, one and only one of numbers n and $n+1$ is even.

Pf. $2 \mid n \Rightarrow \exists k \in \mathbb{Z}, n = 2k \Rightarrow n+1 = 2k+1 \Rightarrow 2 \nmid n+1$

k

$\Longrightarrow 2 \mid n+1$ as $k+1 \in \mathbb{Z}$. $\blacksquare$

Ex. Find all possible $a \in \mathbb{R}$ such that

$$\exists! \, x \in \mathbb{R}, \quad x^2 - 2x + a^2 = 0$$

Solution. $x^2 - 2x + a^2 = 0 \iff x^2 - 2x + 1 = 1 - a^2$

$\iff (x-1)^2 = 1 - a^2$

$\iff x - 1 = \pm\sqrt{1-a^2}$

So there is a unique solution $\iff 1 - a^2 = 0$

$\iff a = 1$ or -1.

$\blacksquare$