Ex. Is this a function?

$$f: \mathbb{R}^{\geq 0} \to \mathbb{R}, \quad f(x) = y \text{ if } y^2 = x.$$

Answer. No, because it does NOT assign a unique element of codomain $\mathbb{R}$ to, let's say, 1 as we have $(\pm 1)^2 = 1$.

Remark. A function $f: X \to Y$ assigns a unique element $f(a)$ of codomain Y to every element a of domain.

Ex. Find all $a \in \mathbb{R}$ such that the following is a function:

$$f(x) = \begin{cases} x-1 & x \geq 2 \\ -x+a^2 & x \leq 2 \end{cases}$$

Solution. Since $f(2)$ should be well-defined, we need to have $2-1 = -2+a^2$. So $a^2 = 3$. Therefore

$$a = \pm\sqrt{3}.$$

Composition of functions

$$\left.\begin{array}{l} f: A \to B \\ g: B \to C \end{array}\right\} \Rightarrow$$ The composite $g \circ f$ of f and g is

$$g \circ f: A \to C,$$

$$(g \circ f)(a) = g(f(a)).$$

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$
$g: \mathbb{R} \to \mathbb{R}^{\geq 0}$, $g(x) = x^2$ $\Big\} \Rightarrow g \circ f: \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) = (x+1)^2$

Ex. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 1/x$.

What is $f \circ f$?

Answer. It is NOT defined as co-domain of $f \neq$ domain of f.

Ex. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $f(x) = 1/x$,

What is $f \circ f$?

Answer. $f \circ f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$,

$$(f \circ f)(x) = f(1/x) = \frac{1}{1/x} = x.$$

So $f \circ f$ is the identity function $I_{\mathbb{R} \setminus \{0\}}$ on $\mathbb{R} \setminus \{0\}$.

- For any set A, $I_A: A \to A$, $I_A(x) = x$ is the identity function on A.

Ex. For any function $f: X \to Y$, we have

$$f \circ I_X = f = I_Y \circ f.$$

Ex. Complete the missing information, if any.

f

$f_1(x) = x+1, \quad f_2(x) = \sqrt{x}, \quad$ and

$$f_2 \circ f_1 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}, \quad f_2 \circ f_1(x) = \sqrt{x+1}.$$

<u>Solution</u>. Domains and co-domains of f_1 and f_2 are missing.

$$\begin{tikzcd}
\mathbb{R}^{\geq 0} \arrow[r, "f_2"] \arrow[rr, bend right, "f_2 \circ f_1"'] & ? \arrow[r, "f_1"] & \mathbb{R}
\end{tikzcd}$$

? should contain <u>image of f_2</u> $\Rightarrow \mathbb{R}^{\geq 1} \subseteq ?$

f_1 should assign values on ? $\Rightarrow ? \subseteq \mathbb{R}^{\geq 0}$

So ? can be any set $\mathbb{R}^{\geq 1} \subseteq A \subseteq \mathbb{R}^{\geq 0}$.

For instance, we can say

$$f_1 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}, \quad f_1(x) = x+1$$

$$f_2 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}, \quad f_2(x) = \sqrt{x}. \qquad \blacksquare$$

Image of a function $f : X \longrightarrow Y$ is

$$\operatorname{Im}(f) = \{ f(x) \mid x \in X \}.$$

So it is a subset of the co-domain Y.

<u>Ex.</u> What is the image of $f : \mathbb{R}^{\geq 2} \longrightarrow \mathbb{R}, \; f(x) = x^3$?

<u>Solu.</u>
$$\left.\begin{array}{l}
\cdot\ 2 \leq x \Rightarrow 8 \leq x^3 \\
\cdot\ 8 \leq y \Rightarrow y = \left(\sqrt[3]{y}\right)^3 \text{ and } 2 \leq \sqrt[3]{y}
\end{array}\right\} \Rightarrow \operatorname{Im}(f) = \mathbb{R}^{\geq 8}.$$

- Find $\operatorname{Im}(f)$ where $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$.

<u>Solution</u>. • $\forall x\in\mathbb{R}$, $x^2\geq 0 \implies \operatorname{Im}(f)\subseteq\mathbb{R}^{\geq 0}$. (I)

• $y\in\mathbb{R}^{\geq 0} \implies y=(\sqrt{y})^2=f(\sqrt{y}) \implies y\in\operatorname{Im}(f)$

So $\mathbb{R}^{\geq 0}\subseteq\operatorname{Im}(f)$. (II)

(I), (II) $\implies \operatorname{Im}(f)=\mathbb{R}^{\geq 0}$.