

Ex. Is this a function?

$$f: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}, \quad f(x) = y \text{ if } y^2 = x.$$

Answer. No, because it does NOT assign a unique element of codomain  $\mathbb{R}$  to, let's say, 1 as we have  $(\pm 1)^2 = 1$ .

Remark. A function  $f: X \rightarrow Y$  assigns a unique element  $f(a)$  of codomain  $Y$  to every element  $a$  of domain.

Ex. Find all  $a \in \mathbb{R}$  such that the following is a function:

$$f(x) = \begin{cases} x-1 & x \geq 2 \\ -x+a^2 & x \leq 2 \end{cases}$$

Solution. Since  $f(2)$  should be well-defined, we need to have

$$2-1 = -2+a^2. \text{ So } a^2 = 3. \text{ Therefore}$$

$$a = \pm\sqrt{3}.$$

### Composition of functions

$f: A \rightarrow B$   
 $g: B \rightarrow C$

$\} \Rightarrow$  The composite  $g \circ f$  of  $f$  and  $g$  is

$g \circ f: A \rightarrow C,$

$$(g \circ f)(a) = g(f(a)).$$

$$\begin{array}{l} \underline{\text{Ex.}} \quad f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x+1 \\ \quad \quad g: \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}, \quad g(x) = x^2 \end{array} \left. \vphantom{\begin{array}{l} f \\ g \end{array}} \right\} \Rightarrow \begin{array}{l} g \circ f: \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}, \\ (g \circ f)(x) = (x+1)^2 \end{array}$$

Ex.  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, \quad f(x) = 1/x.$

What is  $f \circ f$ ?

Answer. It is NOT defined as co-domain of  $f \neq$  domain of  $f$ .

Ex.  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}, \quad f(x) = 1/x,$

What is  $f \circ f$ ?

Answer.  $f \circ f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\},$

$$(f \circ f)(x) = f(1/x) = \frac{1}{1/x} = x.$$

So  $f \circ f$  is the identity function  $I_{\mathbb{R} \setminus \{0\}}$  on  $\mathbb{R} \setminus \{0\}$ .

• For any set  $A$ ,  $I_A: A \rightarrow A, \quad I_A(x) = x$  is the identity function on  $A$ .

Ex. For any function  $f: X \rightarrow Y$ , we have

$$f \circ I_X = f = I_Y \circ f.$$

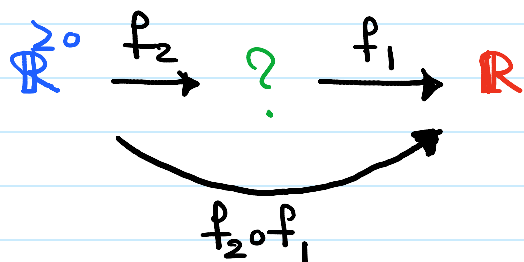
Ex. Complete the missing information, if any.

$f$

$$f_1(x) = x+1, \quad f_2(x) = \sqrt{x}, \quad \text{and}$$

$$f_2 \circ f_1: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}, \quad f_2 \circ f_1(x) = \sqrt{x+1}.$$

Solution. Domains and co-domains of  $f_1$  and  $f_2$  are missing.



? should contain image of  $f_2$   $\Rightarrow \mathbb{R}^{\geq 1} \subseteq ?$

$f_1$  should assign values on ?  $\Rightarrow ? \subseteq \mathbb{R}^{\geq 0}$

So ? can be any set  $\mathbb{R}^{\geq 1} \subseteq A \subseteq \mathbb{R}^{\geq 0}$ .

For instance, we can say

$$f_1: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}, \quad f_1(x) = x+1$$

$$f_2: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}, \quad f_2(x) = \sqrt{x}. \quad \blacksquare$$

Image of a function  $f: X \rightarrow Y$  is

$$\text{Im}(f) = \{f(x) \mid x \in X\}.$$

So it is a subset of the co-domain  $Y$ .

Ex. What is the image of  $f: \mathbb{R}^{\geq 2} \rightarrow \mathbb{R}, f(x) = x^3$ ?

Solu.  $2 \leq x \Rightarrow 8 \leq x^3$   
 $8 \leq y \Rightarrow y = (\sqrt[3]{y})^3$  and  $2 \leq \sqrt[3]{y}$   $\Rightarrow \text{Im}(f) = \mathbb{R}^{\geq 8}$ .

• Find  $\text{Im}(f)$  where  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ .

Solution. •  $\forall x \in \mathbb{R}, x^2 \geq 0 \Rightarrow \text{Im}(f) \subseteq \mathbb{R}^{\geq 0}$ . (I)

•  $y \in \mathbb{R}^{\geq 0} \Rightarrow y = (\sqrt{y})^2 = f(\sqrt{y}) \Rightarrow y \in \text{Im}(f)$

So  $\mathbb{R}^{\geq 0} \subseteq \text{Im}(f)$ . (II)

(I), (II)  $\Rightarrow \text{Im}(f) = \mathbb{R}^{\geq 0}$ .