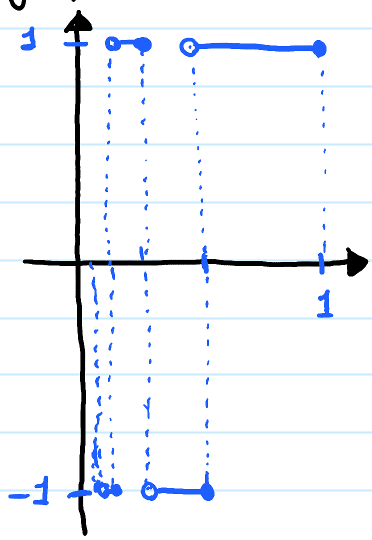
Friday, November 6, 2015 3:30 PM

1. Let $f:(0,1] \to [-1,1]$, for any $k \in \mathbb{Z}^{\geq 0}$,

$$f(x) = (-1)^k \quad \text{if} \quad \frac{1}{2^{k+1}} < x \leq \frac{1}{2^k}.$$

So its graph looks like:



Prove that $\lim_{x \to 0^+} f(x)$ does NOT exist.

(<u>Hint</u>: It is similar to the example that we did in class: $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does NOT exist.)

2. (a) Prove or disprove: $\forall x \in \mathbb{R}, \left((\forall \varepsilon > 0, |x| \leq \varepsilon) \Rightarrow x = 0 \right)$.

(b) Prove or disprove: $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \left(|x| \leq \varepsilon \Rightarrow x = 0 \right)$.

3. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

4. (a) Find all possible $a \in \mathbb{R}$ such that

$$\exists x \in \mathbb{R}, \; x^2 - 2x + a^2 = 0.$$

(b) Find all possible $a \in \mathbb{R}$ such that

$$\exists!\, x \in \mathbb{R},\ x^2 - 2x + a^2 = 0.$$

5. Prove that there are 2^n functions $f : \{1, 2, \cdots, n\} \to \{0, 1\}$.

(Hint. Use induction on n.)

6. For $A \subseteq X$, the characteristic function $\mathbb{1}_A$ of A is

$$\mathbb{1}_A : X \to \{0, 1\},\quad \mathbb{1}_A(x) = \begin{cases} 1 & x \in A. \\ 0 & x \in X \setminus A. \end{cases}$$

(a) Prove that $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$.

(b) Prove that $\mathbb{1}_{A^c} + \mathbb{1}_A = \mathbb{1}_X$.