Problem set 7

Monday, November 16, 2015 12:27 AM

1. Let F_n be the set of functions $f:\{1,2,\dots,n\}\to\{0,1\}$.

For instance, $F_1=\{f_1,f_2\}$ where $f_1:\{1\}\to\{0,1\}$, $f_1(1)=0$ and $f_2:\{1\}\to\{0,1\}$, $f_2(1)=1$.

Let $\theta: F_n \to P(\{1,2,\dots,n\})$,

$$\theta(f)=\{k\in\mathbb{Z} \mid 1\le k\le n,\ f(k)=1\},$$

and $\psi: P(\{1,2,\dots,n\}) \to F_n$,

$$\psi(A)=\mathbb{1}_A$$

(the characteristic function of A as a subset of $\{1,\dots,n\}$. (for its definition look at the previous problem set.))

(a) Find $\theta(f)$ where $f:\{1,2,3,4,5\}\to\{0,1\}$,

$f(1)=1,\ f(2)=0,\ f(3)=0,\ f(4)=1,\ f(5)=0$.

(b) Prove that $\psi\circ\theta=I_{F_n}$ for any $n\in\mathbb{Z}^+$.

(c) Prove that $\theta\circ\psi=I_{P(\{1,\dots,n\})}$ for any $n\in\mathbb{Z}^+$.

(Remark. (b) and (c) show that ψ and θ are 1-1 and onto. So F_n and $P(\{1,\dots,n\})$ have the same number of elements.)

2. Let $f: X \to Y$, $g: Y \to Z$ be two functions.

Prove that, if f and g are injective, then $g \circ f$ is injective.

3. Let $f: X \to Y$, $g: Y \to Z$ be two functions.

Prove that, if f and g are surjective, then $g \circ f$ is surjective.

4. (a) Prove that ($g \circ f$ injective $\Rightarrow$ f injective).

(b) Prove or disprove ($g \circ f$ injective $\Rightarrow$ g injective.)

5. (a) Prove that ($g \circ f$ surjective $\Rightarrow$ g surjective.)

(b) Prove or disprove ($g \circ f$ surjective $\Rightarrow$ f surjective).

Added example for problem 1.

Let $n=2$. Then $F_2 = \{g_1, g_2, g_3, g_4\}$ where

$g_1 : \{1,2\} \to \{0,1\}$, $g_1(1) = 1$, $g_1(2) = 1$

$g_2 : \{1,2\} \to \{0,1\}$, $g_2(1) = 1$, $g_2(2) = 0$

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$g_4 : \{1,2\} \longrightarrow \{0,1\}$, $g_4(1) = 0$, $g_4(2) = 0$.

So $\theta : \{g_1, g_2, g_3, g_4\} \longrightarrow \{\{\}, \{1\}, \{2\}, \{1,2\}\}$,

$\theta(g_1) = \{k \in \{1,2\} \mid g_1(k) = 1\} = \{1, 2\}$

$\theta(g_2) = \{1\}$

$\theta(g_3) = \{2\}$

$\theta(g_4) = \{\}$.