Solution 2.

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1. Prove that for any integer n one and exactly one of the numbers n and $n+1$ is even.

(b) Prove that, for any integer n,

$$n(n+1) \text{ is even.}$$

Solution (a) n is even $\Rightarrow n=2k$ for some integer k

$\Rightarrow n+1=2k+1$

$\Rightarrow n+1$ is odd (we proved in class)

n is odd $\Rightarrow n=2k+1$ for some integer k

$\Rightarrow n+1=2k+2=2(k+1)$

$\Rightarrow n+1$ is even as $k+1$ is integer.

(b) Case-by-Case

Case 1. n is even.

$2 \mid n \Rightarrow n=2k$ for some integer

$\Rightarrow n(n+1)=2k(n+1)$

$\Rightarrow 2 \mid n(n+1)$ as $k(n+1)$ is integer.

Case 2. n is odd.

n is odd $\underset{\text{(part a)}}{\Rightarrow}$ $n+1$ is even $\Rightarrow n+1=2k'$ for some integer k'

$\Rightarrow n(n+1)=2k'n$

$\Rightarrow 2 \mid n(n+1)$ as $k'n$ is integer.

Second Proof. We prove by contradiction. Suppose to the contrary that $n(n+1)$ is odd for some integer n.

$\Rightarrow n$ and $n+1$ are odd (in class we proved mn is odd $\Leftrightarrow m$ and n are odd.)

$\Rightarrow n$ and $n+1$ are odd (in class we proved
mn is odd $\Leftrightarrow$ m and n are odd.)

which contradicts part (a). ■

2. Prove that $201x - 9y = 2$ has no integer solution

Solution. Suppose to the contrary that it has integer solutions x_0, y_0.
$\Rightarrow 2 = 201x_0 - 9y_0 = 3(67x_0 - 3y_0) \Rightarrow 3 \mid 2$ as $67x_0 - 3y_0$ is integer $\Rightarrow 3 \le 2$ which is a contradiction (in class we proved For non-zero integers a, b, $a \mid b \Rightarrow |a| \le |b|$.) ■

3. Prove that for any positive real numbers x, y, z

$$\sqrt{\frac{x^2+y^2+z^2}{3}} \ge \frac{x+y+z}{3}.$$

Proof. Backward argument

$$\sqrt{\frac{x^2+y^2+z^2}{3}} \ge \frac{x+y+z}{3} \Leftarrow \frac{x^2+y^2+z^2}{3} \ge \left(\frac{x+y+z}{3}\right)^2$$

($|a| \le |b| \Leftrightarrow a^2 \le b^2$.)

$$\Leftarrow \frac{x^2+y^2+z^2}{3} \ge \frac{x^2+y^2+z^2+2xy+2xz+2yz}{9}$$

$$\Leftarrow 3x^2+3y^2+3z^2 \ge x^2+y^2+z^2+2(xy+xz+yz)$$

$$\Leftarrow 2(x^2+y^2+z^2) \ge 2(xy+xz+yz)$$

$$\Leftarrow x^2+y^2+z^2 \ge xy+xz+yz$$ (is proved in class.) ■

4. Determine if the following statements are true or not.

Justify your answer.

(a) For any integers m and n,

$$6 \mid mn \Rightarrow 6 \mid m \vee 6 \mid n.$$

(b) For any integers m and n,

(b) For any integers m and n,

$$6|m \vee 6|n \Rightarrow 6|mn.$$

(c) For any integers m and n,

$$3|mn \Rightarrow 3|m \vee 3|n.$$

<u>Solution</u>. (a) False; let $m=3$ and $n=2$.

Then $6 \mid (3)(2)$ and $6 \nmid 3$ and $6 \nmid 2$.

(If $6|3$, then $6 \leq 3$ which is a contradiction.
If $6|2$, then $6 \leq 2$ 〃 〃 〃 〃.)

(b) <u>Case-by-case</u>.

<u>Case 1.</u> $6|m$.

$6|m \Rightarrow m=6k$ for some integer k

$\Rightarrow mn = 6kn$

$\Rightarrow 6|mn$ as kn is an integer.

<u>Case 2.</u> $6|n$

By a similar argument as in case 1, we have $6|mn$.

(c) We prove by contradiction. Suppose to the contrary that for <u>some</u> integers m and n we have $3|mn$, $3 \nmid m$, $3 \nmid n$. So by the hint there are integers k, ℓ s.t. $m = 3k \pm 1$ and $n = 3 \pm 1$.

Hence $mn = (3k \pm 1)(3\ell \pm 1)$

$$= 9k\ell \pm 3\ell \pm 3k \pm 1$$

$$= 3(3k\ell \pm \ell \pm k) \pm 1. \quad \text{I}$$

$$= 3(3kl \pm l \pm k) \pm 1 . \quad \text{I}$$

$3 \mid mn \Rightarrow mn = 3k'$ for some integer k' II

I, II $\Rightarrow 3k' - 3(3kl \pm l \pm k) = \pm 1 .$

$\Rightarrow \pm 3(k' - kl \pm l \pm k) = 1$

$\Rightarrow 3 \mid 1 \Rightarrow 3 \leq 1$ which is a contra. ■

5. Let d be an integer more than 1, and a_1, a_2, b_1, and b_2 are integers. Suppose $d \mid a_1 - a_2$ and $d \mid b_1 - b_2$.

Prove that $d \mid (a_1 + b_1) - (a_2 + b_2)$.

and $d \mid a_1 b_1 - a_2 b_2$.

Proof. $d \mid a_1 - a_2 \Rightarrow$ for some integer k,

$a_1 - a_2 = dk$ I

$d \mid b_1 - b_2 \Rightarrow$ for some integer l,

$b_1 - b_2 = dl .$ II

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$$

$$= dk + dl \quad \text{(by I, II)}$$

$$= d(k + l)$$

$\Rightarrow d \mid (a_1 + b_1) - (a_2 + b_2)$ as $k + l$ is an integer.

$$a_1 b_1 - a_2 b_2 = (a_1 - a_2) b_1 + a_2 (b_1 - b_2)$$

$$= dkb_1 + a_2 dl \quad \text{(by I, II)}$$

$$= d(kb_1 + a_2 l)$$

$\Rightarrow d \mid a_1b_1 - a_2b_2$ as $kb_1 + a_2\ell$ is an integer. ■