1. Let F_n be the set of functions $f:\{1,2,\dots,n\}\to\{0,1\}$

For instance, $F_1=\{f_1,f_2\}$ where $f_1:\{1\}\to\{0,1\}$, $f_1(1)=0$ and $f_2:\{1\}\to\{0,1\}$, $f_2(1)=1$.

Let $\Theta: F_n \to \mathcal{P}(\{1,2,\dots,n\})$,

$$\Theta(f)=\{k\in\mathbb{Z} \mid 1\le k\le n,\ f(k)=1\},$$

and $\Psi:\mathcal{P}(\{1,2,\dots,n\})\to F_n$,

$$\Psi(A)=\mathbb{1}_A$$

(the characteristic function of A as a subset of $\{1,\dots,n\}$. (for its definition look at the previous problem set.))

(a) Find $\Theta(f)$ where $f:\{1,2,3,4,5\}\to\{0,1\}$,

$f(1)=1$, $f(2)=0$, $f(3)=0$, $f(4)=1$, $f(5)=0$.

(b) Prove that $\Psi\circ\Theta=I_{F_n}$ for any $n\in\mathbb{Z}^+$.

(c) Prove that $\Theta\circ\Psi=I_{\mathcal{P}(\{1,\dots,n\})}$ for any $n\in\mathbb{Z}^+$.

Proof. (a) $\Theta(f)=\{k\in\{1,2,3,4,5\} \mid f(k)=1\}$

We check elements of $\{1,2,3,4,5\}$ one-by-one to see if they belong to $\Theta(f)$ or not.

$$\left.\begin{array}{l} f(1)=1 \Rightarrow 1\in\Theta(f) \\ f(2)=0 \Rightarrow 2\notin\Theta(f) \\ f(3)=0 \Rightarrow 3\notin\Theta(f) \\ f(4)=1 \Rightarrow 4\in\Theta(f) \\ f(5)=0 \Rightarrow 5\notin\Theta(f) \end{array}\right\} \Rightarrow \Theta(f)=\{1,4\}.$$

Before we go to the proof parts (b) and (c). Let's get a better

Before we go to the proof parts (b) and (c). Let's get a better understanding of Θ and Ψ.

By the definition of Θ, for a function $f:\{1,2,\dots,n\}\to\{0,1\}$, $\Theta(f)$ is a subset of $\{1,2,\dots,n\}$ and

$\forall k\in\{1,2,\dots,n\}, \quad k\in\Theta(f) \iff f(k)=1.$ I

(and so $k\notin\Theta(f) \iff f(k)=0$.)

By the definition of Ψ, for a subset A of $\{1,2,\dots,n\}$, $\Psi(A)$ is a function $\Psi(A):\{1,2,\dots,n\}\to\{0,1\}$ and

$\forall k\in\{1,2,\dots,n\}, \quad k\in A \iff \Psi(A)(k)=1.$ II

(and so $k\notin A \iff \Psi(A)(k)=0$.)

$\boxed{\text{Part (b)}}$

$$\begin{tikzcd} F_n \arrow[r, "\Theta"] \arrow[rr, bend right, "\Psi\circ\Theta"'] & P(\{1,\dots,n\}) \arrow[r, "\Psi"] & F_n \end{tikzcd}$$

So $\Psi\circ\Theta$ has the same domain and codomain as I_{F_n}. Hence it is enough to show:

$$\forall f\in F_n, \quad \Psi\circ\Theta(f)=f.$$

Both $(\Psi\circ\Theta)(f)$ and f are functions $\{1,\dots,n\}\to\{0,1\}$.

$$\begin{aligned}
\forall k\in\{1,\dots,n\},\quad ((\Psi\circ\Theta)(f))(k)=1 & \\
\iff & \Psi(\Theta(f))(k)=1 \\
\text{(by II)} \iff & k\in\Theta(f) \\
\text{(by I)} \iff & f(k)=1.
\end{aligned}$$

So $\forall k\in\{1,\dots,n\}$, $(\Psi\circ\Theta)(f)(k)=f(k)$

$\Rightarrow (\Psi\circ\Theta)(f)=f.$

$\boxed{\text{Part (c)}}$

$$\begin{tikzcd} P(\{1,2,\dots,n\}) \arrow[r, "\Psi"] \arrow[rr, bend right, "\Theta\circ\Psi"'] & F_n \arrow[r, "\Theta"] & P(\{1,2,\dots,n\}) \end{tikzcd}$$

$\theta \circ \psi$

So $\theta \circ \psi$ has the same domain and codomain as $I_{P(\{1,\dots,n\})}$.

It is enough to show $\forall A \subseteq \{1,2,\dots,n\}$, $\theta \circ \psi(A) = A$.

$\forall k \in \{1,2,\dots,n\}$, $k \in (\theta \circ \psi)(A) = \theta(\psi(A))$

$$\text{I} \quad \iff \quad \psi(A)(k) = 1$$

$$\text{II} \quad \iff \quad k \in A$$

So $(\theta \circ \psi)(A) = A$.

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2. Let $f: X \to Y$, $g: Y \to Z$ be two functions. Prove that, if f and g are injective, then $g \circ f$ is injective.

Proof. $(g \circ f)(x_1) = (g \circ f)(x_2) \implies g(f(x_1)) = g(f(x_2))$

$(g \text{ is injective}) \implies f(x_1) = f(x_2)$

$(f \text{ is injective}) \implies x_1 = x_2$. ■

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3. Let $f: X \to Y$, $g: Y \to Z$ be two functions. Prove that, if f and g are surjective, then $g \circ f$ is surjective.

Proof. We have to prove $\forall z \in Z, \exists x \in X, (g \circ f)(x) = z$.

Since g is surjective, $\exists y \in Y$, $g(y) = z$.

Since f is surjective, $\exists x \in X$, $f(x) = y$. So

$$(g \circ f)(x) = g(f(x)) = g(y) = z. \quad ■$$

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4. (a) Prove that $(g \circ f \text{ injective} \implies f \text{ injective})$.

(b) Prove or disprove ($g \circ f$ injective $\Rightarrow$ g injective.)

Proof. (a) $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow x_1 = x_2 .$$

(b) Not true. We need to give two functions f and g such that $g \circ f$ is injective and g is not injective.

$f : \{1\} \to \{1, 2\}$, $f(1) = 1$

$g : \{1, 2\} \to \{1\}$, $g(1) = g(2) = 1$. NOT injective.

So $g \circ f : \{1\} \to \{1\}$, $(g \circ f)(1) = 1$ is injective.

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5. (a) Prove that ($g \circ f$ surjective $\Rightarrow$ g surjective.)

(b) Prove or disprove ($g \circ f$ surjective $\Rightarrow$ f surjective).

Proof. (a) We have to prove that

$$\forall z \in Z, \exists y \in Y, g(y) = z \quad \circledast$$

Since $g \circ f : X \to Z$ is surjective, $\exists x \in X$, $(g \circ f)(x) = z$.

So $g(f(x)) = z$ and $y = f(x)$ satisfies $\circledast$.

(b) Not true. We have to give functions f and g such that $g \circ f$ is surjective and f is NOT surjective.

The same example as in problem 4.b works.