

1. Let F_n be the set of functions $f: \{1, 2, \dots, n\} \rightarrow \{0, 1\}$

For instance, $F_1 = \{f_1, f_2\}$ where $f_1: \{1\} \rightarrow \{0, 1\}$, $f_1(1) = 0$

and $f_2: \{1\} \rightarrow \{0, 1\}$, $f_2(1) = 1$.

Let $\Theta: F_n \rightarrow P(\{1, 2, \dots, n\})$,

$$\Theta(f) = \{k \in \mathbb{Z} \mid 1 \leq k \leq n, f(k) = 1\},$$

and $\Psi: P(\{1, 2, \dots, n\}) \rightarrow F_n$,

$$\Psi(A) = \underset{A}{\mathbf{1}} \quad (\text{the characteristic function of } A \text{ as a subset of } \{1, 2, \dots, n\}. \text{ (for its definition look at the previous problem set.)})$$

(a) Find $\Theta(f)$ where $f: \{1, 2, 3, 4, 5\} \rightarrow \{0, 1\}$,

$$f(1) = 1, f(2) = 0, f(3) = 0, f(4) = 1, f(5) = 0.$$

(b) Prove that $\Psi \circ \Theta = I_{F_n}$ for any $n \in \mathbb{Z}^+$.

(c) Prove that $\Theta \circ \Psi = I_{P(\{1, \dots, n\})}$ for any $n \in \mathbb{Z}^+$.

Proof. (a) $\Theta(f) = \{k \in \{1, 2, 3, 4, 5\} \mid f(k) = 1\}$

We check elements of $\{1, 2, 3, 4, 5\}$ one-by-one to see if they belong to $\Theta(f)$ or not.

$$f(1) = 1 \Rightarrow 1 \in \Theta(f) \quad \left. \Rightarrow \Theta(f) = \{1, 4\} \right.$$

$$f(2) = 0 \Rightarrow 2 \notin \Theta(f)$$

$$f(3) = 0 \Rightarrow 3 \notin \Theta(f)$$

$$f(4) = 1 \Rightarrow 4 \in \Theta(f)$$

$$f(5) = 0 \Rightarrow 5 \notin \Theta(f)$$

Before we go to the proof parts (b) and (c). Let's get a better

Before we go to the proof parts (b) and (c). Let's get a better understanding of Θ and Ψ .

By the definition of Θ , for a function $f: \{1, 2, \dots, n\} \rightarrow \{0, 1\}$, $\Theta(f)$ is a subset of $\{1, 2, \dots, n\}$ and

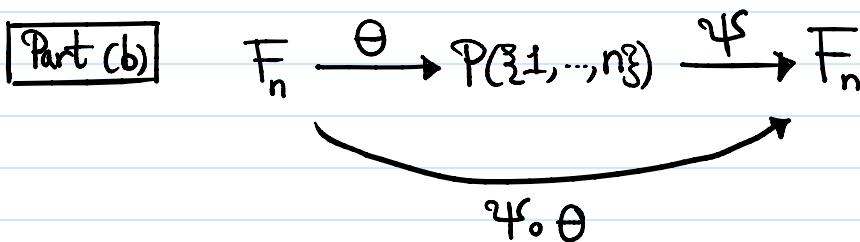
$$\forall k \in \{1, 2, \dots, n\}, \quad k \in \Theta(f) \iff f(k) = 1. \quad \text{(I)}$$

(and so $k \notin \Theta(f) \iff f(k) = 0.$)

By the definition of Ψ , for a subset A of $\{1, 2, \dots, n\}$, $\Psi(A)$ is a function $\Psi(A): \{1, 2, \dots, n\} \rightarrow \{0, 1\}$ and

$$\forall k \in \{1, 2, \dots, n\}, \quad k \in A \iff \Psi(A)(k) = 1.$$

(and so $k \notin A \iff \Psi(A)(k) = 0.$)



So $\Psi \circ \Theta$ has the same domain and codomain as I_{F_n} . Hence it is enough to show:

$$\forall f \in F_n, \quad \Psi \circ \Theta(f) = f.$$

Both $(\Psi \circ \Theta)(f)$ and f are functions $\{1, \dots, n\} \rightarrow \{0, 1\}$.

$$\forall k \in \{1, \dots, n\}, \quad ((\Psi \circ \Theta)(f))(k) = \underline{}$$

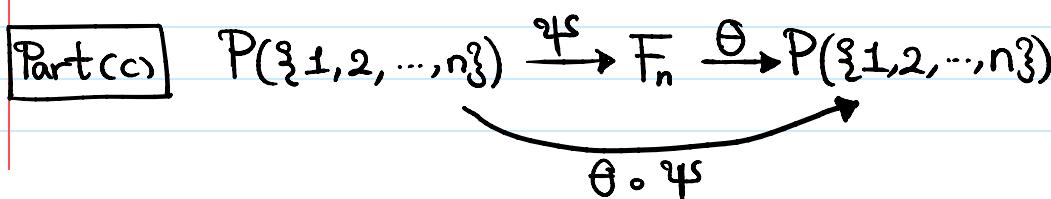
$$\iff \Psi(\Theta(f))(k) = 1$$

$$(\text{by (I)}) \iff k \in \Theta(f)$$

$$(\text{by (I)}) \iff f(k) = 1.$$

$$\text{So } \forall k \in \{1, \dots, n\}, \quad (\Psi \circ \Theta)(f)(k) = f(k)$$

$$\Rightarrow (\Psi \circ \Theta)(f) = f.$$



$$\Theta \circ \Psi$$

So $\Theta \circ \Psi$ has the same domain and codomain as $I_{P(\{1, \dots, n\})}$.

It is enough to show $\forall A \subseteq \{1, 2, \dots, n\}, \Theta \circ \Psi(A) = A$.

$\forall k \in \{1, 2, \dots, n\}, k \in (\Theta \circ \Psi)(A) \iff \Theta(\Psi(A))$

$$\textcircled{I} \iff \Psi(A)(k) = 1$$

$$\textcircled{II} \iff k \in A$$

So $(\Theta \circ \Psi)(A) = A$.

2. Let $f: X \rightarrow Y, g: Y \rightarrow Z$ be two functions.

Prove that, if f and g are injective, then $g \circ f$ is injective.

$$\text{Proof. } (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$

$$(g \text{ is injective}) \Rightarrow f(x_1) = f(x_2)$$

$$(f \text{ is injective}) \Rightarrow x_1 = x_2 \quad \blacksquare$$

3. Let $f: X \rightarrow Y, g: Y \rightarrow Z$ be two functions.

Prove that, if f and g are surjective, then $g \circ f$ is surjective.

Proof. We have to prove $\forall z \in Z, \exists x \in X, (g \circ f)(x) = z$.

Since g is surjective, $\exists y \in Y, g(y) = z$.

Since f is surjective, $\exists x \in X, f(x) = y$. So

$$(g \circ f)(x) = g(f(x)) = g(y) = z \quad \blacksquare$$

4.(a) Prove that ($g \circ f$ injective $\Rightarrow f$ injective).

Lemma: If $y_1 = y_2 \Rightarrow f(y_1) = f(y_2)$

$$\begin{aligned}\text{Proof. (a)} \quad f(x_1) &= f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \\ &\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \\ &\Rightarrow x_1 = x_2.\end{aligned}$$

(b) Not true. We need to give two functions f and g such that $g \circ f$ is injective and g is not injective.

$$f: \{1\} \rightarrow \{1, 2\}, \quad f(1) = 1$$

$$g: \{1, 2\} \rightarrow \{1\}, \quad g(1) = g(2) = 1. \quad \text{NOT injective.}$$

$$\text{So } g \circ f: \{1\} \rightarrow \{1\}, \quad (g \circ f)(1) = 1 \quad \text{is injective.}$$

5.(a) Prove that $(g \circ f \text{ surjective} \Rightarrow g \text{ surjective.})$

(b) Prove or disprove $(g \circ f \text{ surjective} \Rightarrow f \text{ surjective}).$

Proof. (a) We have to prove that

$$\forall z \in Z, \exists y \in Y, \quad g(y) = z \quad \textcircled{*}$$

Since $g \circ f: X \rightarrow Z$ is surjective, $\exists x \in X, (g \circ f)(x) = z$.

So $g(f(x)) = z$ and $y = f(x)$ satisfies $\textcircled{*}$.

(b) Not true. We have to give functions f and g such that $g \circ f$ is surjective and f is NOT surjective.

The same example as in problem 4.b works.