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1. Let $f(x) = 1 + \dfrac{1}{1+\frac{1}{x}} = \dfrac{2x+1}{x+1}$ for positive real number x.

(a) Prove that $f(x)$ is increasing, i.e. for any real numbers x_1 and x_2,

$$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2).$$

For the rest of this problem, let $a_0 = 1$ and, for any non-negative integer n, $a_{n+1} = f(a_n)$. (f is the above function.)

(b) Prove that for any non-negative integer n,

$$a_n \le \frac{1+\sqrt{5}}{2}$$

[Hint. First observe that $f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2}$.]

(c) Prove that $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence, i.e. for any non-negative integer n, $a_n \le a_{n+1}$.

(d) Prove that $\lim\limits_{n\to\infty} a_n = \dfrac{1+\sqrt{5}}{2}$.

[Remark. Using a similar technique one can show that $b_0 = 2 = 1 + \frac{1}{1}$, $b_{n+1} = f(b_n)$ defines a decreasing sequence

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which converges to $\frac{1+\sqrt{5}}{2}$. Altogether we have $1+\cfrac{1}{1+\cfrac{1}{1+\cdots}} = \frac{1+\sqrt{5}}{2}$.

This is an example of a continued fraction.]

2. Prove that for any positive integer n,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Let $b_1 = 1$, $b_{n+1} = 1 + \frac{1}{b_n}$ for any positive integer n.

So we get the following initial terms:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$$

(a) Prove that for any positive integer n,

$$b_n = \frac{F_{n+1}}{F_n},$$

where $F_0, F_1, \ldots$ is the Fibonacci sequence.

(b) Prove that for any positive integer n,

$$b_{n+1} - b_n = \frac{(-1)^{n+1}}{F_n F_{n+1}}.$$

4. (Postage stamp problem) Prove that any postage greater than 34 can be obtained by stamps of denominations 5 and 9.

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[Hint ⓪ You need to show for any integer $n \geq 34$ there are non-negative integers x and y such that

$$n = 5x + 9y.$$

① Use strong induction on n.

② $34 = 5 \times 5 + 9$,

$35 = 5 \times 7$,

$36 = 9 \times 4$,

$37 = 5 \times 2 + 9 \times 3$,

$38 = 5 \times 4 + 9 \times 2$.]

6. Problem 12 from page 54
7. Problem 20 from page 56
8. Problem 21 from page 56