Problem set 8

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1. Prove that, for two non-empty sets A and B,

$$\text{There is a surjection } A \xrightarrow{f} B \iff |B| \leq |A|.$$

2. Prove that, for any real numbers $a<b$, we have that the open interval (a,b) is equipotent to the open interval $(0,1)$.

3. (a) Prove that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$. (You are allowed to use results mentioned in class long ago without proof!)

   (b) Using part (a), show $(0,1) \sim \mathbb{R}$.

4. Prove that, for any two non-empty sets A and B,

$$|A| = |B| \implies |\mathcal{P}(A)| = |\mathcal{P}(B)|.$$

5. (a) Prove that $\{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\}$ is enumerable.

   (Hint. Let $f: \{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\} \to \mathbb{Z}^{+}$,

$$f(\{m_1, \dots, m_k\}) = 2^{m_1} + \dots + 2^{m_k}.\ )$$

   (b) Prove that there is no surjection

$$g: \{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\} \to \mathcal{P}(\mathbb{Z}^{\geq 0}),$$

   where $\mathcal{P}(\mathbb{Z}^{\geq 0})$ is the power set of $\mathbb{Z}^{\geq 0}$. (Hint. Cantor.)

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In exercise 5, you are allowed to use the fact that any positive integer has a unique binary representation, i.e.

$\forall n \in \mathbb{Z}^+, \exists!\ m_1, \dots, m_k \in \mathbb{Z}^{\geq 0}$, $0 \leq m_1 < m_2 < \dots < m_k$

and $n = 2^{m_k} + 2^{m_{k-1}} + \dots + 2^{m_1}$.

6. Determine if the following functions are injective or surjective. Justify your answers.

(a) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f((a,b)) = 3a - 2b$.

(b) Let $A \subseteq X$, and $\ell: P(X) \to P(X)$, $\ell(B) = A \Delta B$. (<u>Hint</u>. What is $\ell \circ \ell(B)$?)

(c) Let Y be a non-empty subset of X, and $r: P(X) \to P(Y)$, $r(B) = Y \cap B$.

7. Let $X_E = \{A \in P(\{1, 2, \dots, n\}) \mid |A| \text{ is even}\}$ and $X_O = \{A \in P(\{1, 2, \dots, n\}) \mid |A| \text{ is odd}\}$.

(a) Show that $\ell_1: X_E \to X_O$, $\ell_1(A) = A \Delta \{1\}$ and $\ell_2: X_O \to X_E$, $\ell_2(A) = A \Delta \{1\}$ are well-defined.

(b) Show that ℓ_1 is the inverse of ℓ_2.

(c) Conclude that $|X_E| = |X_O| = 2^{n-1}$.