Lecture 7: Induction principle

Friday, October 7, 2016 9:06 AM

In the previous class we saw two examples where we sort of understood by creating a kind of method to go forward one step at the time. In order to make that approach formal, we need the induction principle:

> To prove that For any positive integer n, $P(n)$ holds, it is enough to show
>
> (Base of induction) $P(1)$ holds
>
> (The inductive step) For any positive integer k, if $P(k)$ holds, then $P(k+1)$ holds.

Using the induction principle, let's prove the first question in the previous lecture:

Problem. Prove that for any positive integer n,

$$1+3+\cdots+(2n-1)=n^2.$$

Solution. Base case. For $n=1$, LHS$=1$ and RHS$=1^2=1$. ✓

Lecture 7: Induction principle.

Friday, October 7, 2016 9:19 AM

Inductive Step. We have to prove:

For any positive integer k, if $1+3+\cdots+(2k-1)=k^2$, then

$$1+3+\cdots+(2k-1)+(2k+1)=(k+1)^2.$$

So suppose for some positive integer k we have

$1+3+\cdots+(2k-1)=k^2$. Then

$$\underbrace{1+3+\cdots+(2k-1)}_{k^2}+(2k+1)=k^2+(2k+1)$$
$$=(k+1)^2,$$

which proves the inductive step. ■

Let's recall the second question from the previous lecture.

Question. What is $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$?

We start with $a_1=\sqrt{2}$ and realized that the pattern

is given by $a_{n+1}=\sqrt{2+a_n}$; by a "visualization" method

conjectured

① For any positive integer n, $0<a_n<2$

② For any positive integer n, $a_n<a_{n+1}$.

Let's prove these "conjectures" using the induction hypothesis.

Lecture 7: Induction principle

Friday, October 7, 2016 9:32 AM

Lemma 1. Suppose $a_1=\sqrt{2}$, and for any positive integer n $a_{n+1}=\sqrt{2+a_n}$. Prove that for any positive integer n

$$0<a_n<2.$$

Proof. We use induction on n.

Base of induction. We have to check $0<a_1<2$.

. $\sqrt{2}$ is clearly positive.

. $\sqrt{2}<2 \Leftarrow 2<4$ (in the previous lecture we proved that $|x|\le|y| \Leftrightarrow x^2\le y^2$.)

Induction step. For a given positive integer k, we assume $0<a_k<2$. We have to show $0<a_{k+1}<2$.

. $0<a_k \Rightarrow 2<a_k+2 \Rightarrow 0<a_k+2 \Rightarrow 0<\sqrt{a_k+2} \Rightarrow 0<a_{k+1}$.

. $a_k<2 \Rightarrow a_k+2<2+2=4 \Rightarrow \sqrt{a_k+2}<\sqrt{4} \Rightarrow a_{k+1}<2$.

$\boxed{x^2\le y^2 \Leftrightarrow |x|\le|y|}$

∎

Lemma 2. In the above setting, for any positive integer n,

$$a_n<a_{n+1}.$$

Proof. We use induction on n.

Base of induction. We have to show $a_1<a_2$.

Lecture 7: Induction principle

Friday, October 7, 2016 12:49 PM

$$\sqrt{2} < \sqrt{2+\sqrt{2}} \Longleftarrow 2 < 2+\sqrt{2} \Longleftarrow 0 < \sqrt{2} \quad \checkmark.$$

<u>The inductive step.</u> For a give positive integer, we assume $a_k < a_{k+1}$. We have to show $a_{k+1} < a_{k+2}$.

We use <u>backward argument</u>:

$$a_{k+1} < a_{k+2} \Longleftarrow \sqrt{2+a_k} < \sqrt{2+a_{k+1}}$$

$$\Longleftarrow 2+a_k < 2+a_{k+1}$$

$$\Longleftarrow a_k < a_{k+1}$$

which is the induction hypothesis. ■

<u>Theorem</u>. $\lim_{n\to\infty} a_n = 2$, which implies $2 = \sqrt{2+\sqrt{2+\sqrt{\cdots}}}$.

<u>Proof.</u> By Lemma 1, a_n is a bounded sequence.

By Lemma 2, a_n is increasing. Hence $\lim_{n\to\infty} a_n$ exists.

Let $L = \lim_{n\to\infty} a_n$. Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+L}$$

Hence $L^2 = 2+L$ which implies $L^2 - L - 2 = 0$. Therefore $(L-2)(L+1) = 0$. So $L=2$ or $L=-1$. Since $a_n > 0$, $L \geq 0$. Therefore $\lim_{n\to\infty} a_n = 2$. ■