Lecture 8: Induction and the Fibonacci sequence

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In today's lecture we study some of the properties of the Fibonacci sequence.

$F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for any positive integer n.

So it has some similarities with $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

Both of them are recursive, but a_{n+1} needs only $\underline{1}$ information and F_{n+1} needs $\underline{2}$.

Let $v_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$. So $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

Hence $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v_n$.

Now we have a recursive formula which only depends on $\underline{1}$ step back.

<u>Theorem</u>. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then for any positive integer n, we have

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

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Proof. We use induction on n.

Base of induction. For $n=1$, $LHS = A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

$$RHS = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1 \\ 1 & 0 \end{bmatrix} \quad \checkmark .$$

The induction step. Suppose for a given positive integer k we have $A^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$. We have to show

$$A^{k+1} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}.$$

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{k+1}+F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix}$$

by the induction hypothesis

$$= \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}. \blacksquare$$

Corollary 1. For any positive integer n,

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$$

where $F_0, F_1, \ldots$ is the Fibonacci sequence.

Proof. $A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. So

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Hence $F_{n+1} \cdot F_{n-1} - F_n^2 = \det(A^n)$

$$= \det(A)^n = (-1)^n . \quad \blacksquare$$

<u>Corollary 2.</u> For any positive integers m, n, we have

$$F_{n+m} = F_{m+1} F_n + F_m F_{n-1},$$

where $F_0, F_1, \dots$ is the Fibonacci sequence.

<u>Proof.</u> We know that for any matrix A and positive integers m and n we have (why?)

$$A^{m+n} = A^m \cdot A^n .$$

Let's use the above equality for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and apply the main theorem:

$$\begin{bmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{m+1} F_{n+1} + F_m F_n & F_{m+1} F_n + F_m F_{n-1} \\ F_m F_{n+1} + F_{m-1} F_n & F_m F_n + F_{m-1} F_{n-1} \end{bmatrix}$$

Comparing the 12-entries, we get $F_{n+m} = F_{m+1} F_n + F_m F_{n-1}$. $\blacksquare$

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<u>Theorem</u>. For any positive integers m and n, we have

$$m \mid n \implies F_m \mid F_n$$

where $F_0, F_1, F_2, \dots$ is the Fibonacci sequence.

<u>Proof</u>. Let's fix a positive integer m. And let n range through multiples of m. So $n = mk$ where k is a positive integer.

So we can rewrite what we need to prove:

for a given positive integer m,

for any positive integer k, $F_m \mid F_{mk}$.

We use induction on k.

<u>Base of induction</u>. $k=1$. We have to prove $F_m \mid F_m$, which is clear as $F_m = F_m \times 1$.

<u>The induction step</u>. Assume for a give positive integer ℓ we have $F_m \mid F_{m\ell}$. Now we have to show $F_m \mid F_{m(\ell+1)}$.

$$F_{m(\ell+1)} = F_{m\ell+m} = F_{m+1} \cdot F_{m\ell} + F_m \cdot F_{m\ell-1}$$

Corollary 2 applied to $n = m\ell$ and m

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By the induction hypothesis, $F_m \mid F_{m\ell}$. So there is an integer s such that $F_{m\ell} = (F_m)(s)$. So

$$F_{m(\ell+1)} = F_{m+1} \cdot F_m \cdot s + F_m \cdot F_{m\ell-1}$$

$$= F_m \underbrace{\left(F_{m+1} \cdot s + F_{m\ell-1} \right)}_{\text{integer}}$$

Hence $F_m \mid F_{m(\ell+1)}$. ∎

Remark 1. The converse of the above Theorem is also correct:

$$F_m \mid F_n \Longrightarrow m \mid n.$$

Remark 2. A recursive sequence like

$$x_{n+1} = a\, x_n + b\, x_{n-1}$$

is called a