Lecture 15: Limit

Friday, October 28, 2016 9:24 AM

We view basics of game theory only as an auxilary tool to understand propositions with universal and existential quantifiers better:

A proposition of the form $\forall x \in X, \exists y \in Y, P(x,y)$ can be interpreted as a <u>losing game</u>:

For every choice x of the "1st player", the "2nd player" can find a <u>suitable</u> "respond" y. (<u>Suitable</u> means $P(x,y)$ holds for this choice of y.)

A proposition of the form $\exists x \in X, \forall y \in Y, Q(x,y)$ can be interpreted as a <u>winning game</u>:

The "1st player" has a good choice x such that for any "move" (choice of y) of the "2nd player", $Q(x,y)$ is going to hold.

(You will not be asked about games.)

Now we review the ε-δ definition of limit. We interpret it in terms of a losing game.

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<u>Definition</u>. We say $\lim_{x \to a} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \left(0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right).$$

So this can be viewed as a <u>losing game</u>.

No matter how the "1st player" challenges us by choosing $\varepsilon > 0$, we as "2nd players" can find a suitable $\delta > 0$ to meet his challenge: for <u>any</u> <u>x</u> that is <u>δ-close</u> to a we get that <u>$f(x)$</u> is <u>ε-close</u> to L.

<u>Ex.</u> Prove that $\lim_{x \to 2} x^2 = 4$.

<u>Proof</u>. We have to show $\forall \varepsilon > 0, \exists \delta > 0, 0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$.

"1st player makes his move" and gives us $\varepsilon > 0$. Now we, as "2nd players" should think about "our move". We have to find a suitable $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

To find a right move, we use backward argument:

$$|x^2 - 4| < \varepsilon \Leftarrow |x-2||x+2| < \varepsilon$$

We'd like to reach to this conclusion under some control on $|x-2|$ (we are allowed to make this as small as we wish!)

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Let's start with an initial "estimate". Let's say we will definitely choose $\delta \le 1$. (The choice of 1 is fairly flexible. Its main point is for us to be able to get an upper bound for $|x+2|$.)

That means we can assume $|x-2|<1$. So $1<x<3$ and $3<x+2<5$, which implies $|x+2|<5$. Hence

$$|x^2-4|<\varepsilon \Leftarrow |x-2||x+2|<\varepsilon$$

$$\Leftarrow |x-2|<\varepsilon/5 \wedge |x+2|<5$$

$$\Leftarrow |x-2|<\varepsilon/5 \wedge |x-2|<1$$

$$\Leftarrow |x-2|<\min(1,\varepsilon/5).$$

Therefore $\delta=\min(1,\varepsilon/5)$ is a suitable choice. ■

__Ex.__ Prove $\lim_{x\to 2} \sqrt{x}=\sqrt{2}$.

__Proof.__ We have to prove $\forall \varepsilon>0, \exists \delta>0, 0<|x-2|<\delta \Rightarrow |\sqrt{x}-\sqrt{2}|<\varepsilon$.

Again this means for a given $\varepsilon>0$, we should find a suitable $\delta>0$ such that the above implication holds. Again we try to use a backward argument.

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$$|\sqrt{x}-\sqrt{2}|<\varepsilon \Leftarrow |x-2|<\varepsilon\, |\sqrt{x}+\sqrt{2}|$$

So this time we need a lower bound for the auxilary function $|\sqrt{x}+\sqrt{2}|$. And the idea is that when x is fairly close to 2, we expect that $\sqrt{x}+\sqrt{2}$ is fairly close to $2\sqrt{2}$. Hence we should be able to get a lower bound for $|\sqrt{x}+\sqrt{2}|$.

Let's again decide that we choose "our move" $\delta \leq 1$.

Hence $|x-2|<1 \Rightarrow 1<x<3$

$$\Rightarrow 1<\sqrt{x}<\sqrt{3}$$

$$\Rightarrow 1+\sqrt{2}<\sqrt{x}+\sqrt{2}<\sqrt{3}+\sqrt{2}$$

$$\Rightarrow 1<|\sqrt{x}+\sqrt{2}|.$$

Therefore $|\sqrt{x}-\sqrt{2}|<\varepsilon \Leftarrow |x-2|<\varepsilon\, |\sqrt{x}+\sqrt{2}|$

$$\Leftarrow |x-2|<\varepsilon \wedge 1<|\sqrt{x}+\sqrt{2}|$$

$$\Leftarrow |x-2|<\varepsilon \wedge |x-2|<1$$

$$\Leftarrow |x-2|<\min(1,\varepsilon).$$

Thus $\delta=\min(1,\varepsilon)$ is a suitable choice. ∎