Lecture 16: Limit does not exist.

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In the previous lecture we learned the $\varepsilon-\delta$ definition of limit, and saw two examples on how to prove a limit exists. In todays lecture, we will see what it means to say a limit does NOT exist. Along the way we learn how to negate propositions that have both universal and existential quantifiers.

Recall we say $\lim\limits_{x\to a} f(x) = L$ if

$$\forall \varepsilon>0,\ \exists\, \delta>0,\ (0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon)$$

For any $\varepsilon>0$, there is $\delta>0$, such that

if <u>x is δ-close to a</u>, then <u>$f(x)$ is ε-close to L</u>.

Interpreting this statement in terms of games:

You challenge me to get $\underbrace{\varepsilon\text{-close to } L}_{\text{your "move"}}$, I have to find a $\underbrace{\text{suitable } \delta}_{\text{my "move"}}$ to meet your "challenge", i.e. to guarantee that <u>$f(x)$ gets ε-close to L</u> it is <u>enough</u> to choose <u>x δ-close to a</u>.

So it is a "losing game".

To say $\lim\limits_{x\to a} f(x)$ does NOT exist, we have to show

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$\forall L \in \mathbb{R}, \lim_{x\to a} f(x) \neq L.$

[Remark. In calculus, sometimes it is said $\lim_{x\to a} f(x) = +\infty$.

This still means $\lim_{x\to a} f(x)$ does NOT exist, but we are also

adding the reason by saying that the quantity $f(x)$ is

arbitrarily large if x gets closer and closer to a.]

For a given $L \in \mathbb{R}$, what does it mean $\lim_{x\to a} f(x) \neq L$?

To find the negation, one can use the game theory interpretation:

The opposite of a losing game is a winning game. So the first player should be able to find a nice move. In this case, it means s/he should be able to challenge the 2^{nd} player with a suitable $\varepsilon > 0$, so that no move of the 2^{nd} player could meet this challenge: for any $\delta > 0$

$\neg$(if x is δ-close to a, then $f(x)$ is ε-close to L.)

A conditional proposition fails exactly when its hypothesis holds and its conclusion fails. One other thing to which we have to

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pay attention is the implicit universal quantifier for x: the above implication is supposed to be true for any $x \in \mathbb{R}$.

So we need to understand

$$\neg\,(\forall x \in \mathbb{R},\ x \text{ is } \delta\text{-close to } a \Rightarrow f(x) \text{ is } \varepsilon\text{-close to } L.)$$

For some $x \in \mathbb{R}$, x is δ-close to a and $f(x)$ is NOT ε-close to L.

In mathematical language we write it

$$\exists x \in \mathbb{R},\ 0<|x-a|<\delta \ \wedge\ |f(x)-L|<\varepsilon.$$

So overall we have

$$\boxed{\begin{aligned} \lim_{x\to a} f(x) \text{ does NOT exist} &\iff \forall L \in \mathbb{R},\ \lim_{x\to a} f(x) \neq L. \\ &\iff \forall L \in \mathbb{R},\ \exists \varepsilon>0,\ \forall \delta>0,\ \neg\,(0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon) \\ &\iff \forall L \in \mathbb{R},\ \exists \varepsilon>0,\ \forall \delta>0,\ \exists x,\ 0<|x-a|<\delta \ \wedge\ |f(x)-L| \geq \varepsilon. \end{aligned}}$$

Often one can use the following templates to write negation of statements involving quantifiers, but one has to be careful about paranthesis.

$$\neg\,(\forall x \in A,\ P(x)) \equiv \exists x \in A,\ \neg P(x).$$

$$\neg\,(\exists x \in A,\ P(x)) \equiv \forall x \in A,\ \neg P(x).$$

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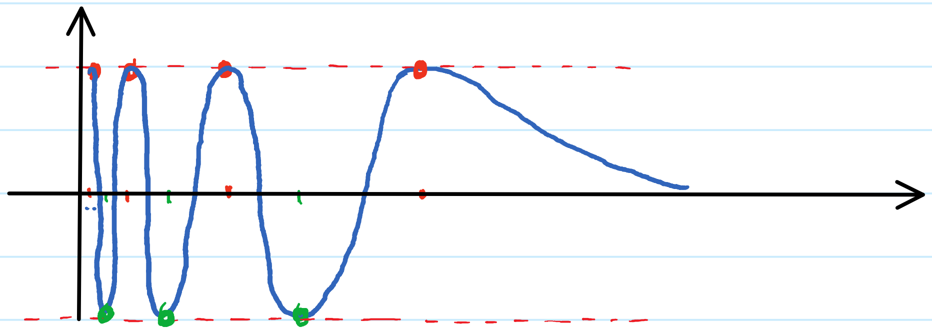
Problem. Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does NOT exist.

We start with trying to visualize the problem by looking at the graph $y = \sin\left(\frac{1}{x}\right)$



As you can see the blue curve can get close to any point on the segment $[-1,1]$ in the y-axis. (The set which consists of the mentioned segment and graph of $\sin(1/x)$ is an interesting set. In topology you will learn that this set is connected, but it is not path-connected.)

We focus on the points at "top" and "bottom". I.e. we will find two sequences x_n^+ and x_n^- with the following properties both x_n^+ and x_n^- get closer and closer to zero; for any n, $\sin(1/x_n^+) = 1$ and $\sin(1/x_n^-) = -1$.

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Let's see how having these sequences is sufficient to deduce that

$\lim_{x\to 0} \sin(1/x)$ does not exist.

Assume to the contrary that $\lim_{x\to 0} \sin(1/x) = L$. So

for any $\varepsilon > 0$, there is $\delta > 0$ such that

if x is δ-close to 0, then $\sin(1/x)$ is ε-close to L.

Since $x_n^{\pm}$ are getting closer and closer to 0, eventually they get δ-close to 0. Hence $\sin(1/x_n^{\pm})$ are ε-close to L.

Therefore both 1 and -1 are ε-close to L. Thus L is ε-close to 1 and ε-close to -1. But, for $\varepsilon \le 1$, there is no number which is both ε-close to 1 and ε-close to -1.

This gives us a contradiction.

Here is the formal proof:

Step 1. There is a sequence x_n^+ of numbers such that

ⓐ x_n^+ gets closer and closer to 0. I.e.

$\forall \delta > 0, \exists N \in \mathbb{R}, n > N \Rightarrow |x_n^+| < \delta$.

ⓑ $\sin(1/x_n^+) = 1$.

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Proof of Step 1. We start with part (b) and use a backward argument:

$$\sin(1/x_n^+) = 1 \Leftarrow \frac{1}{x_n^+} = \frac{\pi}{2} + 2n\pi$$

$$\Leftarrow x_n^+ = \frac{1}{\frac{\pi}{2} + 2n\pi}.$$

To get part (a), we start with a given $\delta > 0$ and again use backward argument to find a suitable N.

$$|x_n^+| < \delta \Leftarrow \frac{1}{\frac{\pi}{2} + 2n\pi} < \delta$$

$$\Leftarrow \frac{1}{2n\pi} < \delta$$

$$\Leftarrow \frac{1}{2\pi\delta} < n.$$

($N = \frac{1}{2\pi\delta}$ is a suitable choice.)

Step 2. There is a sequence x_n^- such that

(a) x_n^- gets closer and closer to 0.

$\forall \delta > 0, \exists N > 0, n \geq N \Rightarrow |x_n^-| < \delta.$

(b) $\sin(1/x_n^-) = -1.$

Proof of step 2. It is similar to the proof of Step 1.

$$\sin(1/x_n^-) = -1 \Leftarrow \frac{1}{x_n^-} = -\frac{\pi}{2} + 2n\pi \Leftarrow x_n = \frac{1}{-\frac{\pi}{2} + 2n\pi}.$$

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$$|x_n^-| < \delta \Leftarrow \frac{1}{-\frac{\pi}{2}+2n\pi} < \delta$$

$$\Leftarrow \frac{1}{2(n-1)\pi} < \delta$$

$$\Leftarrow \frac{1}{2\pi\delta} < n-1 \Leftarrow \frac{1}{2\pi\delta}+1 < n$$

(So, for $\delta>0$, $N = \frac{1}{2\pi\delta}+1$ is a suitable choice.)

<u>Finishing the proof.</u> Suppose to the contrary $\lim_{x\to 0} \sin(1/x) = L$.

In particular, there is $\delta_0 > 0$ such that

if $0<|x|<\delta_0$, then $\sin(1/x)$ is $\frac{1}{2}$-close to L. (I)

By Step 1 and Step 2, there is N such that

$$n \geq N \Rightarrow 0 < |x_n^{\pm}| < \delta_0 \quad \text{(II)}$$

Hence, by (I), (II),

$$n \geq N \Rightarrow \sin\left(\frac{1}{x_n^+}\right) \text{ and } \sin\left(\frac{1}{x_n^-}\right) \text{ are } \frac{1}{2}\text{-close to } L$$

$$\Rightarrow \left|\sin\left(\frac{1}{x_n^+}\right) - L\right| < \frac{1}{2} \text{ and } \left|\sin\left(\frac{1}{x_n^-}\right) - L\right| < \frac{1}{2}$$

$$\Rightarrow |1-L| < \frac{1}{2} \text{ and } |-1-L| < \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} < L < \frac{3}{2} \text{ and } -3/2 < L < -1/2$$

which is a contradiction. ∎

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The key idea in the above proof which can be used in similar problems is the following:

To show $\lim_{x\to a} f(x)$ does not exist it is enough to find two sequences x_n^+ and x_n^- such that

(a) x_n^+ and x_n^- get closer and closer to a. I.e.

$\forall \varepsilon>0, \exists N>0, \ n\geq N \Rightarrow |x_n^{\pm}-a|<\varepsilon$.

(we say $\lim_{n\to\infty} x_n^{\pm}=a$.)

(b) $f(x_n^+)$ gets closer and closer to L_1;

$f(x_n^-)$ gets closer and closer to L_2;

And $L_1\neq L_2$.

(I.e. $\forall\varepsilon>0, \exists N>0, \ n\geq N \Rightarrow |f(x_n^+)-L_1|<\varepsilon$ and $|f(x_n^-)-L_2|<\varepsilon$.)

This is part of your homework assignment. To see how useful this is let's use it to show the following:

Problem. Let $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q} \quad \text{(rational)}\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q} \quad \text{(irrational)}.\end{cases}$

Prove that, for any $a\in\mathbb{R}$, $\lim_{x\to a} f(x)$ does NOT exist.

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<u>Sketch of a proof</u> To show this it is enough to notice that any real number <u>a</u> can be approximated by rational numbers x_n^+ and irrational numbers x_n^-. So

- $\lim_{n\to\infty} x_n^{\pm} = a$
- $f(x_n^+) = 1$ and $f(x_n^-) = 0$ for any n.

Hence by the above mentioned property $\lim_{x\to a} f(x)$ does NOT exist.