Lecture 19: Examples on image and graph of functions

Friday, November 4, 2016 9:21 AM

Ex. Is there a function $f: \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Im}(f) = \mathbb{Z}$?

Solution. Yes, there are lots of such functions. For instance

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases} \quad \blacksquare$$

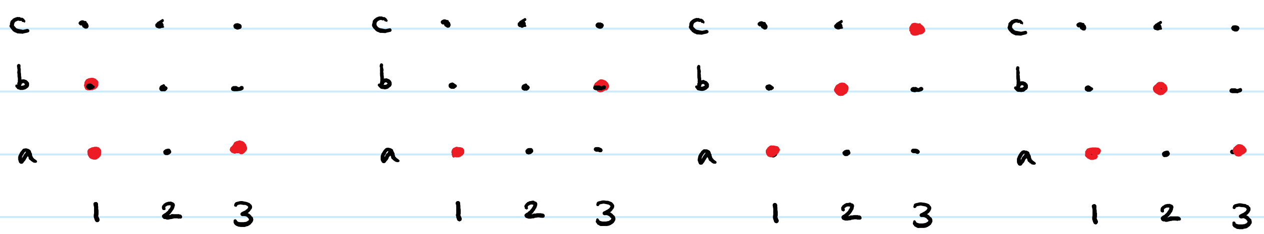
Ex. Is there a function $f: \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Im}(f) = \mathbb{R} \setminus \mathbb{Z}$?

Solution. Yes, again there are lots of such functions. For instance:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 1/2 & \text{if } x \in \mathbb{Z}. \end{cases}$$

can be any non-integer number.

Ex. Which one of the following diagrams represent graph of a function? In each case say whether function is surjective or not?



No, it does NOT assign a unique element to 1

No, it does NOT assign any element to 2

Yes, and it is surjective

Yes, but it is NOT surjective. c is NOT in the image

Lecture 19: Examples of graph; injective functions

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- In graph of a function every "vertical line" intersects the graph in one and exactly one point.

Ex. Suppose $G_f = \{(1,1), (2,3), (4,1)\}$ is graph of a surjective function. Find its domain and codomain.

Solution. First components give us the domain of f and the 2nd components give us the image of f. Since f is surjective we have that codomain $= \mathrm{Im}(f)$. So

domain $= \{1, 2, 4\}$ and codomain $= \{1, 3\}$. ■

Definition. A function $f: X \to Y$ is called injective or one-to-one or 1-1 if

$$\forall x_1, x_2 \in X, \left(f(x_1) = f(x_2) \Rightarrow x_1 = x_2\right).$$

Definition. A function $f: X \to Y$ is called bijective if it is both injective and bijective.

Ex. In each case determine whether the given function is injective, surjective, or bijective.

Lecture 19: Injective, bijective functions

Sunday, November 6, 2016 4:34 PM

(a) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x+1$.

(b) $f:\mathbb{R}^{>0}\to\mathbb{R}$, $f(x)=x^2$.

(c) $f:(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$, $f(x)=\tan(x)$.

Solution. (a) It is a bijection.

Why is it injective? $\forall x_1,x_2\in\mathbb{R}$, $(f(x_1)=f(x_2)\overset{?}{\Rightarrow} x_1=x_2)$

$f(x_1)=f(x_2) \Rightarrow x_1+1=x_2+1 \Rightarrow x_1=x_2$.

Why is it surjective? We have to show

$$\forall y\in\mathbb{R},\ \exists x\in\mathbb{R},\ y=f(x).$$

So for any $y\in\mathbb{R}$, we have to find $x\in\mathbb{R}$ such that $y=x+1$.

We notice $y=x+1 \Leftarrow x=y-1$ and $y-1\in\mathbb{R}$, which gives us the above claim.

(b) It is injective, but not surjective.

Why is it injective? $\forall x_1,x_2\in\mathbb{R}^+$, $(f(x_1)=f(x_2)\overset{?}{\Rightarrow} x_1=x_2)$

$f(x_1)=f(x_2) \Rightarrow x_1^2=x_2^2 \Rightarrow |x_1|=|x_2|$

Since $x_1,x_2\in\mathbb{R}^+$, we have $x_1=|x_1|$ and $x_2=|x_2|$ $\Bigg\} \Rightarrow x_1=x_2$

Why is it not surjective? For any $x\in\mathbb{R}^+$, $x^2>0$. So there is no

$x \in \mathbb{R}^{>0}$ such that $-1 = f(x)$, which implies $-1 \notin \mathrm{Im}(f)$.

Therefore $\mathrm{Im}(f) \neq$ the codomain of f which is $\mathbb{R}$.

[What is $\mathrm{Im}(f)$? Claim: $\mathrm{Im}(f) = \mathbb{R}^+$.

To show this claim, we need to show $\mathrm{Im}(f) \subseteq \mathbb{R}^+$ and

$\mathbb{R}^+ \subseteq \mathrm{Im}(f)$.

Why is $\mathrm{Im}(f) \subseteq \mathbb{R}^+$? We have to show $y \in \mathrm{Im}(f) \Rightarrow y \in \mathbb{R}^+$.

$y \in \mathrm{Im}(f) \Rightarrow \exists x \in \mathbb{R}^+, y = f(x) \Rightarrow \exists x > 0, y = x^2$

$\Rightarrow y > 0 \Rightarrow y \in \mathbb{R}^+$.

Why is $\mathbb{R}^+ \subseteq \mathrm{Im}(f)$? We have to show $y \in \mathbb{R}^+ \Rightarrow y \in \mathrm{Im}(f)$.

(Backward argument) $y \in \mathrm{Im}(f) \Leftarrow \exists x \in \mathbb{R}^+, y = f(x)$

$\Leftarrow \exists x > 0, \; y = x^2 \Leftarrow (\sqrt{y} > 0$ and $(\sqrt{y})^2 = y) \Leftarrow y > 0$.]

(c) It is a bijection.

In class, I used graph of tan x to convey the idea of a

proof: As you can see, any horizontal line intersects the graph in one and exactly one point.

Lecture 19: Injection, surjection, bijection

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Here is a more formal proof using theorems from calculus:

- Function $f:(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$, $f(x)=\tan x$ is a differentiable function and

$$f'(x)=\frac{1}{\cos^2 x}\geq 1 \quad \text{for any } x\in(-\tfrac{\pi}{2},\tfrac{\pi}{2}).$$

(Using mean value theorem, if $x_1<x_2$, then

$\exists y_0$, $x_1<y_0<x_2$ and $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(y_0)\geq 1$.

In particular, $f(x_2)-f(x_1)\geq x_2-x_1$). So

if $x_2>x_1$, then $f(x_2)>f(x_1)$. Therefore f is injective.

We also know $\lim_{x\to\pi/2^+}\tan x=+\infty$ and $\lim_{x\to\pi/2^-}\tan x=-\infty$. Since tan is continuous, by intermediate value theorem we have $\mathrm{Im}(\tan)=\mathbb{R}$.

(Recall. If $f:[a,b]\to\mathbb{R}$ is continuous, $f(a)<f(b)$, and $f(a)\leq y_0\leq f(b)$, then there exists $a\leq x_0\leq b$ such that $f(x_0)=y_0$. (This is called Intermediate value theorem).