Lecture 20: Injection, surjection, composition

Monday, November 7, 2016 9:06 AM

Theorem. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions.

(a) If $g\circ f$ is injective, then f is injective.

(b) If $g\circ f$ is surjective, then g is surjective.

(c) If f and g are injective, then $g\circ f$ is injective.

(d) If f and g are surjective, then $g\circ f$ is surjective.

Proof. (a) We have to show $\forall x_1, x_2 \in X,\ f(x_1)=f(x_2) \overset{?}{\Longrightarrow} x_1=x_2$.

$$f(x_1)=f(x_2) \Longrightarrow g(f(x_1))=g(f(x_2))$$

$$\Longrightarrow (g\circ f)(x_1) = (g\circ f)(x_2)$$

$$\Longrightarrow x_1=x_2 \quad \text{since } g\circ f \text{ is injective.}$$

(b) We have to show $\forall z\in Z, \exists y\in Y,\ g(y)=z$. $\circledast$

We know $g\circ f$ is surjective. So

$$\forall z\in Z,\ \exists x\in X,\ (g\circ f)(x)=z, \text{ which implies}$$
$$\forall z\in Z,\ \exists x\in X,\ g(f(x))=z.$$

For a given $z\in Z$, let $x\in X$ be such that $g(f(x))=z$

then $y=f(x)\in Y$ and $g(y)=z$ which implies $\circledast$.

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(c) We have to show $\forall x_1, x_2 \in X$, $(g\circ f)(x_1) = (g\circ f)(x_2) \Rightarrow x_1 = x_2$.

$$(g\circ f)(x_1) = (g\circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow f(x_1) = f(x_2) \quad \text{since } g \text{ is injective}$$

$$\Rightarrow x_1 = x_2 \quad \text{since } f \text{ is injective.}$$

(d) We have to show $\forall z \in Z$, $\exists x \in X$, $(g\circ f)(x) = z$, which means $g(f(x)) = z$ for some $x \in X$.

We go "one step at a time":

Since g is surjective, for some $y \in Y$ we have $g(y) = z$. Choose such y and call it y_0.

Since f is surjective, for some $x \in X$ we have $f(x) = y_0$. Choose such x and call it x_0.

So we have $g(y_0) = z$ and $f(x_0) = y_0$. Therefore

$(g\circ f)(x_0) = g(f(x_0)) = g(y_0) = z$, as we wished. ∎

<u>Corollary</u>. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions. If $g\circ f$ is a bijection, then f is injective and g is surjective.

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__Corollary__. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ are two functions.

If $g \circ f = I_X$, then f is injective and g is surjective.

__Question__. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ are two functions.

If $g \circ f = I_X$, can we conclude that f and g are bijections?

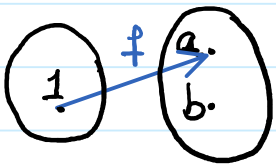
__Answer__. No, we cannot. There are lots of examples. Here are two:

① Let $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = x\,\vec{e}_1 = (x, 0)$ embedding of $\mathbb{R}$ into $\mathbb{R}^2$ as "the x-axis"

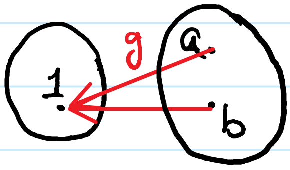
and $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x$ projection onto the x-axis (and parametrizing the x-axis.)

Then $g \circ f: \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) = g(x, 0) = x$.

② Let $X = \{1\}$, $Y = \{a, b\}$

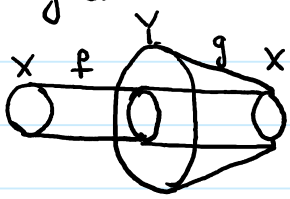


$f(1) = a$, $g(a) = g(b) = 1$.



Then $(g \circ f)(1) = 1$.

f is NOT surjective and g is NOT injective. ■



Lecture 20: invertible functions

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<u>Definition</u>. A function $X \xrightarrow{f} Y$ is called <u>invertible</u> if

$\exists\, g: Y \to X$ and $\exists\, h: Y \to X$ such that

$$g \circ f = I_X \qquad \text{and} \qquad f \circ h = I_Y .$$

$$\begin{tikzcd}
Y \arrow[d, "h"'] \arrow[dr, "f \circ h = I_Y"] & \\
X \arrow[r, "f"] \arrow[dr, "g \circ f = I_X"'] & Y \arrow[d, "g"] \\
& X
\end{tikzcd}$$

(Such a g is called a left inverse, and such an h is called a right inverse.)

In the next lecture we will prove:

<u>Theorem</u>. $f: X \to Y$ is invertible if and only if f is a bijection.