Lecture 22: Bijections

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In the previous lecture we proved two lemmas:

<u>Lemma 1</u>. Suppose $X \xrightarrow{f} Y$ is a function.

f is injective $\iff$ f has a left inverse.

<u>Lemma 2</u>. Suppose $X \xrightarrow{f} Y$ is a function.

f is surjective $\iff$ f has a right inverse.

Using these Lemmas we prove

<u>Theorem</u>. Suppose $X \xrightarrow{f} Y$ is a function.

f is bijective $\iff$ f is invertible.

<u>Proof</u>. f is bijective $\iff \begin{cases} f \text{ is injective} \overset{\text{(Lemma 1)}}{\iff} f \text{ has a left inverse} \\ f \text{ is surjective} \underset{\text{(Lemma 2)}}{\iff} f \text{ has a right inverse} \end{cases}$

$\left.\begin{array}{l} f \text{ has a left inverse} \\ f \text{ has a right inverse} \end{array}\right\} \iff f$ is invertible. ■

<u>Lemma</u>. If g is a left inverse of $f: X \to Y$ and h is a right inverse of f, then $g = h$.

<u>Proof</u>. Consider $g \circ f \circ h$. We have $g \circ f \circ h = g \circ (f \circ h) = g \circ I_Y = g$ and $g \circ f \circ h = (g \circ f) \circ h = I_X \circ h = h$. So $g = h$. ■

<u>Theorem</u>. Suppose $X \xrightarrow{f} Y$ is a function.

f is a bijection $\iff$ there is a unique $g: Y \to X$,

$$g \circ f = I_X \text{ and } f \circ g = I_Y$$

(Such $Y \xrightarrow{g} X$ is called the inverse of f and it is denoted by $f^{(-1)}$.)

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Proof. ($\Rightarrow$) We have to prove two things ① existence of such function ② uniqueness of such function.

① Existence. Since f is a bijection, by the previous theorem, f is invertible. So f has a left inverse g and a right inverse h. By the above lemma, $h=g$. So $g\circ f = I_X$ and $f\circ g = I_Y$.

② Uniqueness. Suppose both $g_1, g_2 : Y \to X$ satisfy the above conditions. So g_1 is a left inverse of f and g_2 is a right inverse of f. Hence, by the above lemma, $g_1 = g_2$, which shows the uniqueness of such function. ■

Theorem (a) If f is a bijection, then f is the inverse of f. And so $f^{(-1)}$ is a bijection.

(b) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two bijections, then $g\circ f : X \to Z$ is a bijection. Moreover,

$$(g\circ f)^{(-1)} = f^{(-1)} \circ g^{(-1)}.$$

Proof. (a) By the definition of the inverse function $f^{(-1)}$, we have $f^{(-1)}\circ f = I_X$ and $f\circ f^{(-1)} = I_Y$. Hence $(f^{(-1)})^{(-1)} = f$. And so by the previous theorem $f^{(-1)}$ is a bijection.

(b) We show that $(g\circ f)\circ(f^{(-1)}\circ g^{(-1)}) = I_Z$

and $(f^{(-1)}\circ g^{(-1)})\circ(g\circ f) = I_X$.

This implies that $g\circ f$ has an inverse and

$$(g\circ f)^{(-1)} = f^{(-1)}\circ g^{(-1)}.$$

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Now, by the previous theorem, we can deduce that $g \circ f$ is a bijection.

$$(g\circ f)\circ(f^{(-1)}\circ g^{(-1)}) = g\circ I_Y \circ g^{(-1)} = g\circ g^{(-1)} = I_Z$$

$$(f^{(-1)}\circ g^{(-1)})\circ(g\circ f) = f^{(-1)}\circ I_Y \circ f = f^{(-1)}\circ f = I_X. \quad \blacksquare$$

<u>Definition</u> Two sets A and B are called <u>equipotent sets</u>, and we write $A\sim B$ if there is a bijection $f: A\to B$.

<u>Lemma.</u> For any non-empty sets A, B, and C, we have

1. $A\sim A$.
2. $A\sim B \Rightarrow B\sim A$.
3. $\left.\begin{array}{l} A\sim B \\ B\sim C \end{array}\right\} \Rightarrow A\sim C$.

<u>Proof</u> 1. $I_A: A\to A$ is a bijection.

2. If $A\xrightarrow{f} B$ is a bijection, then $B\xrightarrow{f^{(-1)}} A$ is a bijection.

3. If $A\xrightarrow{f} B$ and $B\xrightarrow{g} C$ are bijections, then $A\xrightarrow{g\circ f} C$ is a bijection. $\blacksquare$

Based on our intuition of cardinality of finite sets we have:

<u>Theorem</u>. Suppose A and B are two non-empty finite sets. Then $A\sim B \Leftrightarrow |A| = |B|$.

In fact a bit stronger results are true:

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<u>Theorem</u>. Suppose X and Y are non-empty finite sets and $X \xrightarrow{f} Y$ is a function. Then

$$f \text{ is injective} \Rightarrow |X| \leq |Y|.$$

The contra-positive form of the above theorem is called pigeonhole principle.

$$\boxed{|X| > |Y| \Rightarrow \exists\, x_1, x_2 \in X,\ x_1 \neq x_2 \wedge f(x_1) = f(x_2).}$$

Alternatively: If there are n pigeons, m pigeonholes and $n > m$, then at least two pigeons share a pigeonhole.

Later we will see some of its applications.