Lecture 23: Enumerable and countable

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In the previous lecture we defined the notion of equipotent sets: $A \sim B \iff$ There exists a bijection $A \xrightarrow{f} B$.

We have discussed:

- $A \sim A$; $A \sim B \Rightarrow B \sim A$; $\left.\begin{array}{l} A \sim B \\ B \sim C \end{array}\right\} \Rightarrow A \sim C$.
- For two non-empty finite sets A and B,
$$A \sim B \iff |A| = |B|.$$

In particular, if B is finite, $A \subseteq B$, and $A \sim B$, then $A = B$.

$\boxed{Q}$ What if B is NOT finite?

Ex. (Hilbert's hotel) $\mathbb{Z}^+ \sim \mathbb{Z}^{\geq 0}$.

(In Hilbert's hotel, we have roo 1, room 2, ... (infinitely many rooms). All of them are occupied. Let's say guest number i is in the ith room. A new guest arrives; let's call her guest number 0. Can we make a room available for her?)

Proof. We have to construct a bijection $f: \mathbb{Z}^+ \to \mathbb{Z}^{\geq 0}$.

Let $f(k) = k-1$ for any $k \in \mathbb{Z}^+$. It is easy to see that f is a bijection. For instance, you

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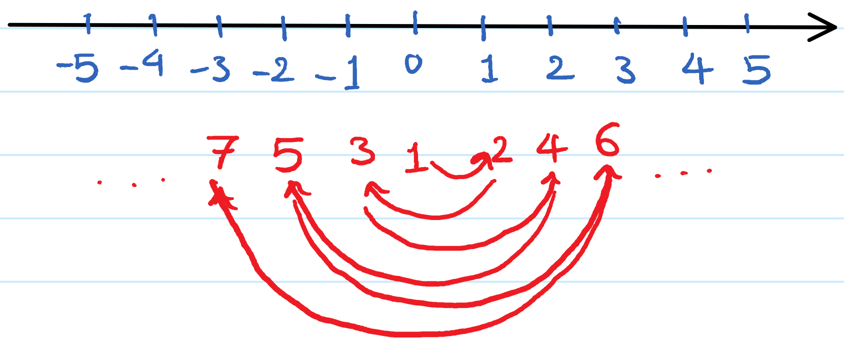
can check that $g: \mathbb{Z}^{\geq 0} \to \mathbb{Z}^+$, $g(k) = k+1$ is an inverse of f. ∎

Definition. • A set X is called <u>enumerable</u> if $X \sim \mathbb{Z}^+$.

• A set X is called <u>countable</u> if X is either <u>finite</u> or it is <u>enumerable</u>.

<u>Ex.</u> $\mathbb{Z}$ is enumerable.

<u>Proof.</u> "We have to enumerate elements of $\mathbb{Z}$."



This picture suggests the following functions:

$$f: \mathbb{Z}^+ \to \mathbb{Z}, \quad f(n) = \begin{cases} -k & \text{if } n = 2k+1, \\ k & \text{if } n = 2k. \end{cases}$$

and

$$g: \mathbb{Z} \to \mathbb{Z}^+, \quad g(n) = \begin{cases} 2n & n > 0, \\ -2n+1 & n \leq 0. \end{cases}$$

Check that f is well-defined and f is an inverse of g. ∎

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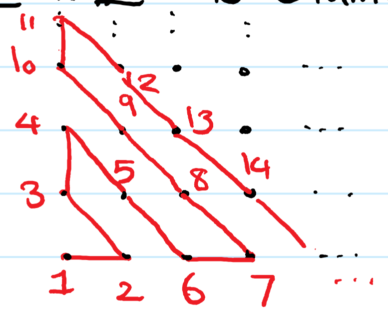
Though writing the details of the above proof might be a bit tricky, the whole idea is in the mentioned "labeling" or "enumerating" of elements of $\mathbb{Z}$:

To show a set A is <u>enumerable</u> it is enough to present <u>a method</u> of labelling A's elements by numbers $1, 2, 3, \dots$ in a way that for sure <u>all</u> the elements of A get labelled at some point. (and only once).

<u>Ex.</u> $\mathbb{Z}^+ \times \mathbb{Z}^+$ is enumerable.

<u>Proof.</u>



Clearly this red path passes through all the points of $\mathbb{Z}^+ \times \mathbb{Z}^+$ once and exactly once. So we get a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+$. ■

<u>Lemma</u>. $(A_1 \sim A_2 \text{ and } B_1 \sim B_2) \Rightarrow A_1 \times B_1 \sim A_2 \times B_2$ for any non-empty sets $A_1, A_2, B_1,$ and B_2.

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Proof. $A_1 \sim A_2 \Rightarrow \exists\ A_1 \xrightarrow{f} A_2$ which is bijective, and

$B_1 \sim B_2 \Rightarrow \exists\ B_1 \xrightarrow{g} B_2$ which is bijective.

Let $A_1 \times B_1 \xrightarrow{h} A_2 \times B_2$, $h(a_1, b_1) = (f(a_1), g(b_1))$. Since f and g are bijective, they have inverses $f^{(-1)}$ and $g^{(-1)}$. Now let $A_2 \times B_2 \xrightarrow{h'} A_1 \times B_1$, $h'(a_2, b_2) = (f^{(-1)}(a_2), g^{(-1)}(b_2))$.

Then
$$
\begin{aligned}
(h \circ h')(a_2, b_2) &= h(f^{(-1)}(a_2), g^{(-1)}(b_2)) \\
&= (f(f^{(-1)}(a_2)), g(g^{(-1)}(b_2))) \\
&= (a_2, b_2),
\end{aligned}
$$

and similarly you can see $(h' \circ h)(a_1, b_1) = (a_1, b_1)$. So h' is an inverse of h. Hence h is a bijection, which implies

$$A_1 \times B_1 \sim A_2 \times B_2. \quad \blacksquare$$

Corollary If A and B are enumerable, then $A \times B$ is enumerable.

Proof. A and B are enumerable $\Rightarrow \left\{ \begin{array}{l} A \sim \mathbb{Z}^+ \\ B \sim \mathbb{Z}^+ \end{array} \right\} \Rightarrow A \times B \sim \mathbb{Z}^+ \times \mathbb{Z}^+$ (by Lemma).

$\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$. So $A \times B \sim \mathbb{Z}^+$, and so $A \times B$ is enumerable. $\blacksquare$