Lecture 24: Review of enumerable sets

Friday, November 18, 2016 3:31 PM

Recall. We say A is equipotent to B, write $A \sim B$, if there is a bijection $A \xrightarrow{f} B$. We have proved:

- $A \sim A$
- $A \sim B \Rightarrow B \sim A$
- $\left.\begin{array}{l} A \sim B \\ B \sim C \end{array}\right\} \Rightarrow A \sim C.$

Definition. Cardinality of A, denoted by $|A|$, is a concept such that $|A| = |B|$ exactly when $A \sim B$.

- The same way that you got positive integers as an abstraction of enumerating objects: getting an abstract concept 3 instead of 3 apples, 3 books, etc.

Def. Cardinality of $\mathbb{Z}^+$ is denoted by the hebrew letter $\aleph_0$ (aleph). So we write $\mathbb{Z}^+ = \aleph_0$.

Def (a) A set X is called enumerable if $X \sim \mathbb{Z}^+$.

(b) A set X is called countable if either X is finite or X is enumerable.

Observation. X is enumerable exactly when $|X| = \aleph_0$.

In the previous lecture, you saw Hilbert's hotel which implies

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$|\mathbb{Z}^{\geq 0}| = |\mathbb{Z}^+| = \aleph_0$. In fact you saw $|\mathbb{Z}| = |\mathbb{Z}^+| = \aleph_0$.

You further saw that $|\mathbb{Z}^+ \times \mathbb{Z}^+| = |\mathbb{Z}^+| = \aleph_0$.

Another Lemma that was proved in the previous lecture was

<u>Lemma</u>.

$$\left.\begin{array}{l} |A_1| = |A_2| \\ |B_1| = |B_2| \end{array}\right\} \Rightarrow |A_1 \times B_1| = |A_2 \times B_2|.$$

<u>Corollary</u>. If $|A| = |B| = \aleph_0$, then $|A \times B| = \aleph_0$.

<u>Proof</u>. $|A| = |\mathbb{Z}^+|$ and $|B| = |\mathbb{Z}^+|$. So

$$|A \times B| = |\mathbb{Z}^+ \times \mathbb{Z}^+| \underset{\boxed{\text{previous result}}}{=} |\mathbb{Z}^+| = \aleph_0. \qquad \blacksquare$$

<u>Definition</u>. We say $|A| \leq |B|$ if there is an <u>injection</u> $f: A \to B$. We say $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$ (which means there is no bijection $A \xrightarrow{f} B$, but there is an injection $A \xrightarrow{g} B$.)

<u>Remark</u>. By <u>Pigeonhole principle</u>, the above definition is consistent with the usual inequality of numbers when A and B are finite sets.

Lecture 24: Cantor's theorem

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$\boxed{Q}$ Is there any uncountable set? I.e. is there a set X such that $|X| > \aleph_0$.

The following theorem of Cantor gives us much more:

<u>Theorem</u>. For any non-empty set X, there is NO surjection $f: X \to P(X)$, where $P(X)$ is the power set of X.

<u>Corollary</u>. $P(\mathbb{Z}^+)$ is uncountable.

<u>Proof of corollary</u>. For any $n \in \mathbb{Z}^+$, $\{n\} \in P(\mathbb{Z}^+)$. So $P(\mathbb{Z}^+)$ is not finite. By Cantor's theorem there is no bijection from $\mathbb{Z}^+$ to $P(\mathbb{Z}^+)$. So $P(\mathbb{Z}^+)$ is NOT enumerable.

Therefore $P(\mathbb{Z}^+)$ is NOT enumerable. ■

<u>Proof of Theorem.</u> Suppose to the contrary that there is a surjection $f: X \to P(X)$. Let

$$A = \{x \in X \mid x \notin f(x)\}.$$

Since f is surjective, $\exists x_0 \in X$, $f(x_0) = A$.

<u>Case 1.</u> $x_0 \in A$. Then $x_0 \notin f(x_0) = A$ which is a contradiction.

<u>Case 2.</u> $x_0 \notin A$. Then $x_0 \in f(x_0) = A$ which is a contradiction. ■

Lecture 24: Continuum hypothesis

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As a corollary of Cantor's theorem, we have:

Cor. For any non-empty set X, $|X| < |P(X)|$.

Warning. To prove this, it is NOT enough to show $n < 2^n$ for any positive integer n. The above corollary is true for infinite sets as well as finite sets.

Proof. ① For any non-empty set X, $f: X \to P(X)$, $f(x) = \{x\}$ is an injection: $f(x_1) = f(x_2) \Rightarrow \{x_1\} = \{x_2\} \Rightarrow x_1 = x_2$.

So $|X| \leq |P(X)|$.

② By Cantor's theorem there is no bijection from X to $P(X)$, and so $|X| \neq |P(X)|$. ■

$\boxed{Q}$ Is there a set A such that $|\mathbb{Z}^+| < |A| < |P(\mathbb{Z}^+)|$?

P. Cohen proved that it is undecidable, which means both this statement and its negation are compatible with the axioms of set theory. Continuum hypothesis asserts that such set does NOT exist.

Def. Following the finite case, we denote $|P(X)|$ by $2^{|X|}$.

Lecture 24: Cantor's diagonal argument

Tuesday, November 15, 2016 12:15 AM

$\boxed{Q}$ Is $\mathbb{R}$ countable?

<u>Answer</u> is No: $\aleph_0 < |\mathbb{R}|$. In fact one can show that $|\mathbb{R}| = 2^{\aleph_0}$.

Cantor proved this using a trick, now known as Cantor's diagonal argument. This proof is based on the following property of real numbers that we accept without proof:

<u>Lemma</u> Any real number x has a unique representation of the form: $x = n + 0.d_1 d_2 \cdots$ where $n, d_i \in \mathbb{Z}, 0 \le d_i \le 9$, and d_i's are NOT eventually all 9's.

<u>Remark</u>. The last condition is necessary in order to get uniqueness: $0.99\cdots = 1$.

<u>Theorem</u>. There is no surjection $\mathbb{Z}^+ \xrightarrow{f} \mathbb{R}$.

<u>Proof</u>. Let $f: \mathbb{Z}^+ \to \mathbb{R}$ be a function. We will find a real number which is not in the image of f. To get such number, we use the representations of $f(n)$'s as described in the above lemma:

Lecture 24: Cantor's diagonal argument

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$$f(1) = m_1 + 0.a_{11}a_{12}a_{13}\cdots$$

$$f(2) = m_2 + 0.a_{21}a_{22}a_{23}\cdots$$

$$f(3) = m_3 + 0.a_{31}a_{32}a_{33}\cdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

To get a real number which is <u>not</u> in the image of f, we have to give a real number which is different from all of these numbers $f(1), f(2), \ldots$. It is enough to write the decimal representation of a number which is different from each of the above ones <u>at least in one digit</u>.

Cantor's idea is to look at the diagonal digits and change them:

Let $d_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases}$. In particular, $d_i \neq a_{ii}$ (and $d_i \neq 9$.)

<u>Claim</u> The real number $0.d_1d_2\cdots$ is NOT in the image of f.

<u>Proof of claim</u>. Suppose to the contrary that

$$f(n) = 0.d_1d_2\cdots$$

So $m_n + 0.a_{n1}a_{n2}\cdots = 0.d_1d_2\cdots$; in particular we should have $a_{nn} = d_n$ which is a contradiction.

Hence $\mathrm{Im}(f) \neq \mathbb{R}$, which means f is NOT surjective. ■