HOMEWORK 2 SOLUTIONS

MICHELLE BODNAR

Problem 1

We'll break the proof into two cases.

Case 1: n is even. Then $n = 2k$ for some integer k and we have $n(n+1) = 2(k(n+1))$ so $n(n+1)$ is even.

Case 2: n is odd. Then $n+1$ is even so $n+1 = 2\ell$ for some integer ℓ and we have $n(n+1) = 2(n\ell)$ so $n(n+1)$ is even. $\square$

Problem 2

Let $p > 1$ be a prime such that $p = ab$ for some integer a and b. Since $p \cdot 1 = ab$, we have that $p|(ab)$, so p divides a or p divides b.

Case 1: $p|a$. Then $|p| \leq |a|$. On the other hand, $p = ab$ so $a|p$ which implies that $|a| \leq |p|$. Thus $|p| = |a|$ so we conclude that $p = \pm a$.

Case 2: $p|b$. Then $|p| \leq |b|$. On the other hand, $p = ab$ so $b|p$ which implies that $|b| \leq |p|$. Thus $|p| = |b|$ so we conclude that $p = \pm b$.

In either case, $p = \pm a \vee p = \pm b$. $\square$

Problem 3

Since $d|a$ we can write $a = dk$ for some integer k. Since $a|b$ we can write $b = a\ell$ for some integer ℓ. Then we have $b = a\ell = d(k\ell)$, so $d|b$. $\square$

Problem 4

Since $d|m$ we can write $m = dk$ for some integer k. Multiplying both sides by r gives $rm = d(kr)$ so $d|rm$. Since $d|n$ we can write $n = d\ell$ for some integer ℓ. Multiplying both sides by s gives $sn = d(\ell s)$ so $d|sn$. By Problem 3, d divides their sum $rm + sn$. $\square$

Problem 5

False. Let $a = 2$ and $b = 3$.

Problem 6

Let $n > 1$ be an integer and d be a divisor of n such that $1 < d < n$. Then we can write $n = dk$ for some integer k. If $d \leq \sqrt{n}$ then we're done since we can simply let $d' = d$. Otherwise, $d > \sqrt{n}$.

I claim that $k \leq \sqrt{n}$. To see this, suppose toward a contradiction that $k > \sqrt{n}$. Then we have $n = dk > \sqrt{n}\sqrt{n} = n$, which is impossible. No number can be strictly larger than itself. Thus, $k \leq \sqrt{n}$. Further, since $d < n$ we must have $n = dk < nk$, which implies that $k > 1$. We can now take $d' = k$. □

Problem 7

$$\begin{aligned}
0 \leq (x-y)^2 &\implies 0 \leq x^2 - 2xy - y^2 \\
&\implies 2xy \leq x^2 + y^2 \\
&\implies 4xy \leq x^2 + 2xy + y^2 \\
&\implies 4xy \leq (x+y)^2 \\
&\implies \frac{4x^2y^2}{(x+y)^2} \leq xy \\
&\implies \frac{2xy}{x+y} \leq \sqrt{xy} \\
&\implies \frac{2xy}{x+y}\frac{1/(xy)}{1/(xy)} \leq \sqrt{xy} \\
&\implies \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy}
\end{aligned}$$

Keep in mind that this string of implications depends on the fact that x and y are *positive* real numbers. This is what justifies preserving the direction of inequality when we multiply both sides by xy, for instance. □