HOMEWORK 4 SOLUTIONS

MICHELLE BODNAR

Problem 1

(a) $\varnothing, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\}$.

(b) $\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$.

Problem 2

(a) True. The only solutions to $(x^2-1)^2 = 0$ are ± 1. However, only one of these is strictly greater than 0. Thus, the set given in the problem is $\{\{1\},\{1\}\} = \{1\}$, which has size 1.

(b) False. The emptyset is an element of $\{\varnothing\}$ but it is not an element of $\{1, \{\varnothing\}\}$.

(c) True. $\{x \in \mathbb{R} | x^2 \geq 0\} = \mathbb{R}$ since the square of any real number is nonnegative. Thus, the set contains only two distinct elements.

Problem 3

(a) By definition of symmetric difference,

$$\begin{aligned} A\Delta\varnothing &= (A \cup \varnothing)\backslash(A \cap \varnothing) \\ &= A\backslash\varnothing \\ &= A. \end{aligned}$$

(b) By definition of symmetric difference,

$$\begin{aligned} A\Delta A &= (A \cup A)\backslash(A \cap A) \\ &= A\backslash A \\ &= \varnothing. \end{aligned}$$

(c) Taking the symmetric difference with A on both sides we have

$$\begin{aligned} A\Delta(A\Delta B) &= A\Delta(A\Delta C) \\ (A\Delta A)\Delta B &= (A\Delta A)\Delta C) \text{ since symmetric difference is associative} \\ \varnothing\Delta B &= \varnothing\Delta C \text{ by part (b)} \\ B &= C \text{ by part (a)} \end{aligned}$$

□

Problem 4

First suppose that $1 \in A$. Then $A\Delta\{1\} = (A \cup \{1\})\backslash(A \cap \{1\}) = A\backslash\{1\}$ and we have $|A\backslash\{1\}| = |A| - 1$. Since $|A|$ is even if and only if $|A| - 1$ is odd, the claim holds. Now suppose $1 \notin A$. Then $A\Delta\{1\} = (A \cup \{1\})\backslash(A \cap \{1\}) = (A \cup \{1\})\backslash\varnothing = A \cup \{1\}$, and we have $|A \cup \{1\}| = |A| + 1$. Since

$|A|$ is even if and only if $|A|+1$ is odd, the claim holds. □

Problem 5

(a) First suppose $A \subset B$. To show that $A \cap B = A$ we'll show that $A \cap B \subset A$ and $A \subset A \cap B$.

Let $x \in A \cap B$ be arbitrary. By definition of intersection, $x \in A$ and $x \in B$. In particular, $x \in A$ so we conclude $A \cap B \subset A$.

Now let $y \in A$ be arbitrary. Since $A \subset B$ we must also have $y \in B$. Therefore $y \in A$ and $y \in B$ so $y \in A \cap B$. We conclude that $A \subset A \cap B$. Therefore $A \cap B = A$.

Next, suppose $A \cap B = A$. We need to show $A \subset B$. Let $x \in A$ be arbitrary. Since $A = A \cap B$, we must have $x \in A \cap B$, which implies that $x \in B$. Thus, $A \subset B$. □

(b) Since B and C are treated identically in this problem, it is enough to show that $B \subset C$, and by symmetry this will imply that $C \subset B$, which means $B = C$. Note: 109 student should go ahead and prove both subset inclusions because it's good practice and many of you don't yet fully understand when you can and cannot apply symmetry arguments!

Suppose towards a contradiction that B is not a subset of C. In other words, that there exists some $x \in B$ such that $x \notin C$. There are now two cases to consider:

Case 1: $x \in A$. Then $x \in A \cap B$ but $x \notin A \cap C$, which contradicts the assumption that $A \cap B = A \cap C$.

Case 2: $x \notin A$. Then $x \in A \cup B$ but $x \notin A \cup C$, which contradicts the assumption that $A \cup B = A \cup C$. In either case we arrive at a contradiction, so we conclude that $B \subset C$. By reversing the roles of B and C, and providing an identical argument, we see that $C \subset B$. Thus, $B = C$ as desired. □