Math 109: The final exam.
Instructor: A. Salehi Golsefidy

Name: ....

PID: ....

09/03/2022

1. Write your Name and PID on the front of your exam sheet.
2. No calculators or other electronic devices are allowed during this exam.
3. Show all of your work; no credit will be given for unsupported answers.
4. Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
5. Ask me or a TA when you are unsure if you are allowed to use certain fact or not.
6. Good luck!

Question	Points	Bonus Points	Score
1	10	0	
2	5	0	
3	10	0	
4	10	0	
5	10	0	
6	10	0	
7	25	0	
8	0	15	
Total:	80	15	

1. (10 points) Which one of the following propositional forms is not equivalent to $P \Rightarrow (Q \vee R)$? Justify your answer.

   1. $(P \wedge (\neg Q)) \Rightarrow R$.
   2. $(P \wedge (\neg Q) \wedge (\neg R)) \Rightarrow \perp$, where $\perp$ means contradiction.
   3. $(P \Rightarrow Q) \vee (P \Rightarrow R)$.
   4. $((\neg Q) \wedge (\neg R)) \Rightarrow (\neg P)$.

2. (5 points) Let $a_0 = 0$ and $a_{n+1} := \sqrt{2 + a_n}$. Prove that, for every $n \in \mathbb{Z}^+$, we have $a_n < a_{n+1}$.

3. (10 points) Write the negation of the following proposition (each part has 5 points):

   (a) $\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+, (\forall n \in \mathbb{Z}^+, (n \geq N \Rightarrow |\frac{\sin n}{n}| < \varepsilon))$.

   (b) $\exists x \in (0,1), \forall y \in (0,1), x \leq y$.

4. (10 points) Suppose A and B are two non-empty sets. Prove that there exists an injective function $f : A \to B$ if and only if there exists a surjective function $g : B \to A$.

5. (10 points) Suppose X is a non-empty set. Prove that there is no surjective function from X to $P(X)$ where $P(X)$ is the power set of X.

6. (10 points) Prove that, for $a, b, c \in \mathbb{Z}^+$, if $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

7. For each question give a short answer. You are allowed to use all the results proved in the lectures:

   (a) (5 points) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, f(x, y) := 5x + 7y$. Prove that f is surjective.

   (b) (5 points) Prove that, for every integers a, b, we have $7|ab$ implies that either $7|a$ or $7|b$.

   (c) (3 points) Let A be a subset of X, $A \neq X$, and

$$g : P(X) \to P(X), g(B) = B \cap A.$$

   Is g injective?

(d) (4 points) Give an infinite set which is not enumerable.

(e) (3 points) Find $x \in \mathbb{Z}$ such that $5x \equiv 3 \pmod{56}$.

(f) (5 points) What is the remainder of 20221090903765 divided by 11?

8. (Bonus) Suppose p is a positive irreducible. Suppose a is an integer and $p \nmid a$. For every $x \in \mathbb{Z}$, let $r_p(x)$ be the remainder of x divided by p. Let

$$f : \{0, \ldots, p-1\} \to \{0, \ldots, p-1\}, \quad f(x) := r_p(ax).$$

(a) (5 points (bonus)) Prove that f is a bijection.

(b) (2 points (bonus)) Prove that $\{1, \ldots, p-1\} = \{f(1), \ldots, f(p-1)\}$.

(c) (3 points (bonus)) Prove that $(p-1)! \equiv (p-1)!a^{p-1} \pmod{p}$. (Hint. Notice that $f(x) \equiv ax \pmod{p}$ for every x)

(d) (5 points (bonus)) Prove that $a^{p-1} \equiv 1 \pmod{p}$.