Lecture 04: Odd and even

Tuesday, August 9, 2022 12:06 AM

In the previous lecture we used division algorithm to prove:

for every integer n, n is odd if and only if $n = 2k+1$ for some integer k.

Using this we can understand various properties of odd and even numbers.

For instance, one can show that sum of two odd numbers is even, sum of an odd and an even integer is odd. We leave these statements as exercises. Next we study product of two odd numbers.

<u>Lemma</u>. For every integer a and b, ab is odd if and only if a and b are odd.

The phrases "if and only if", "precisely when", "necessary and sufficient" are used for biconditional propositions. To prove a biconditional proposition one has to show both "directions". For instance, to show the previous lemma, we have to prove

$$ab \text{ is odd} \Rightarrow (a \text{ is odd} \wedge b \text{ is odd})$$

<u>and</u>

$$ab \text{ is odd} \Leftarrow (a \text{ is odd} \wedge b \text{ is odd})$$

To show the 1st claim, we have to prove that

Lecture 04: Odd and even

Tuesday, August 9, 2022 10:10 AM

that (ab is odd $\Rightarrow$ a is odd) and (ab is odd $\Rightarrow$ b is odd.)

Proof. ($\Rightarrow$) To prove that, ab is odd implies a is odd, we show its contrapositive. The contrapositive of this conditional proposition is $\neg$(a is odd) $\Rightarrow$ $\neg$(ab is odd), which means a is even $\Rightarrow$ ab is even.

a is even $\Rightarrow$ $a = 2k$ for some integer k

$\Rightarrow$ $ab = 2(kb)$

$\Rightarrow$ ab is even as kb is an integer.

alternatively we write $kb \in \mathbb{Z}$.

Notice that, if we swape the variables a and b, the hypothesis does not change. Hence the hypothesis has a symmetry and whatever we can prove about a, based on the hypothesis, holds for b as well. In these cases, we say by symmetry we have that

ab is even $\Rightarrow$ b is even.

($\Leftarrow$) Next we want to show that

a and b are odd $\Rightarrow$ ab is odd.

Lecture 04: Prime and irreducible

Tuesday, August 9, 2022 10:20 AM

$$\left.\begin{array}{l} a \text{ is odd} \Rightarrow a = 2k+1 \text{ for some integer } k \\ b \text{ is odd} \Rightarrow b = 2\ell+1 \text{ for some integer } \ell \end{array}\right\} \Rightarrow$$

$$ab = (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1$$

$$= 2(2k\ell + k + \ell) + 1$$

Hence, ab is of the form $2 \times$ an integer $+1$; and so ab is odd. ■

Looking at the contrapositive of both sides of the biconditional proposition given in the previous lemma, we obtain:

<u>Corollary</u>. For every integers a and b,

$$ab \text{ is even} \Leftrightarrow (a \text{ is even or } b \text{ is even})$$

Using mathematical language, we can state the previous corollary as

$$\forall a, b \in \mathbb{Z}, \quad 2 \mid ab \Leftrightarrow (2 \mid a \vee 2 \mid b).$$

This takes us to the following important concepts.

<u>Definition</u>. (1) An integer p is called <u>irreducible</u> if $p \neq 0$, $p \neq 1$, and for integers a, b, $p = ab$ implies $a = \pm 1$ or $b = \pm 1$.

(2) An integer p is called <u>prime</u> if $p \neq 0$, $p \neq 1$, and for integers a, b, $p \mid ab$ implies $p \mid a$ or $p \mid b$

Lecture 04: 2 is prime

Tuesday, August 9, 2022 10:32 AM

Warning. From elementary school you learned about prime numbers. The definition of prime numbers that you have used so far is different from the one we are using in this course. What you called prime is a positive irreducible number in our terminology. As part of your HW assignment, you will show that

$$\text{prime} \Rightarrow \text{irreducible}.$$

The converse of this proposition is true as well, and we will prove it later in this class; and this is referred to as Euclid's lemma.

$$\text{(Euclid's lemma)} \quad \text{irreducible} \Rightarrow \text{prime}.$$

By the definition, you can think about irreducibles as "atoms": integers that cannot be "split" further (cannot be written as a product of integers with smaller absolute value). The concept of prime is more subtle, and it is ultimately connected with unique factorization of integers.

Going back to the last corollary that we proved, we can restate it as $\quad$ 2 is prime.

Lecture 04: Induction principle

Tuesday, August 9, 2022 10:45 AM

Next we learn about <u>the induction principle</u>. We start with an unofficial introduction.

<u>Ex.</u> Find $1+3+5+\cdots+(2n-1)$. (Whenever there is a three dots symbol, there is a pattern which is supposed to be repeated. For instance, in this example, we are adding the first n odd numbers starting from 1.)

<u>Solution</u>. Whenever facing a new problem, it can be a good idea to start with small examples.

For $n=1$, we get 1. 1^2

For $n=2$, we get $1+3=4$. 2^2

For $n=3$, we get $\underbrace{1+3}_{4}+5=9$. 3^2

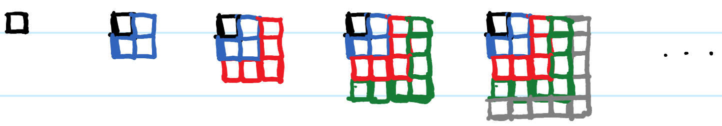
For $n=4$, we get $\underbrace{1+3+5}_{9}+7=16$. 4^2

For $n=5$, we get $\underbrace{1+3+5+7}_{16}+9=25$. 5^2

Based on these, we can form a "guess" (we refer to this as a <u>conjecture</u>.)

$$1+3+\cdots+(2n-1)=n^2.$$

Can we visualize these equations?



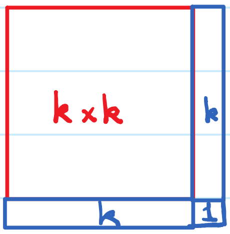
1   3 new   5 new   7 new   9 new

Suppose after k steps we get $k\times k$ square. At the $k+1$ step, we add

Lecture 04: Induction principle

Tuesday, August 9, 2022 11:08 AM

we are adding



2k+1 new squares to get a $(k+1)\times(k+1)$ square and $2k+1$ is the next odd number.

(We sometimes refer to this as a pictorial proof.)

<u>Ex.</u> How can we make sense of $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$? If it does make sense, what number is it?

<u>Initial solution.</u> We start by understanding the three dots. In this case, it means that we have a sequence of numbers given by certain pattern.

$$a_1=\sqrt{2},\quad a_2=\sqrt{2+\sqrt{2}},\quad a_3=\sqrt{2+\sqrt{2+\sqrt{2}}},\quad a_4=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

Based on the above observation, we get the following recursive sequence. $a_1=\sqrt{2}$ and $a_{n+1}=\sqrt{2+a_n}$. The three dots mean what happens to a_n's as we let n go to infinity: find $\lim_{n\to\infty} a_n$.

Lecture 04: Induction principle

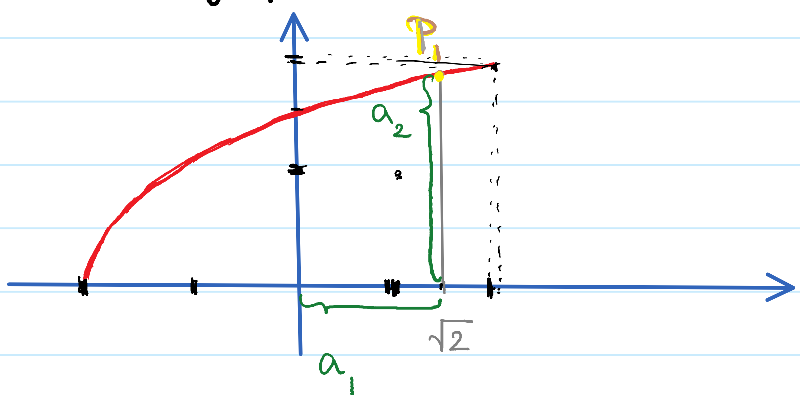
Tuesday, August 9, 2022 3:00 PM

Let $f(x) = \sqrt{2+x}$. Then, for every positive integer k,

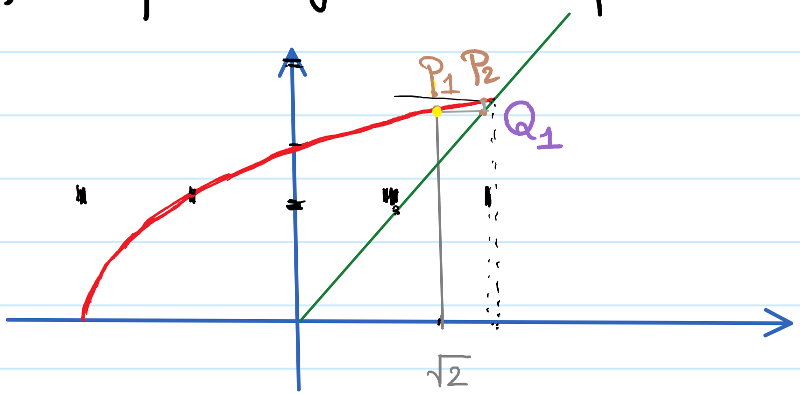
$$a_{k+1} = f(a_k).$$

(Understanding on what happens as we apply a function repeatedly is part of dynamical systems.)

So we start with graph $y=f(x)$ of f.



We notice that $P_1 = (a_1, f(a_1)) = (a_1, a_2)$. Now in order to find $\underline{a_3}$ we need to find $\underline{a_2}$ on the x-axis, (instead of y-axis). Graph of $y=x$ can help us on this.

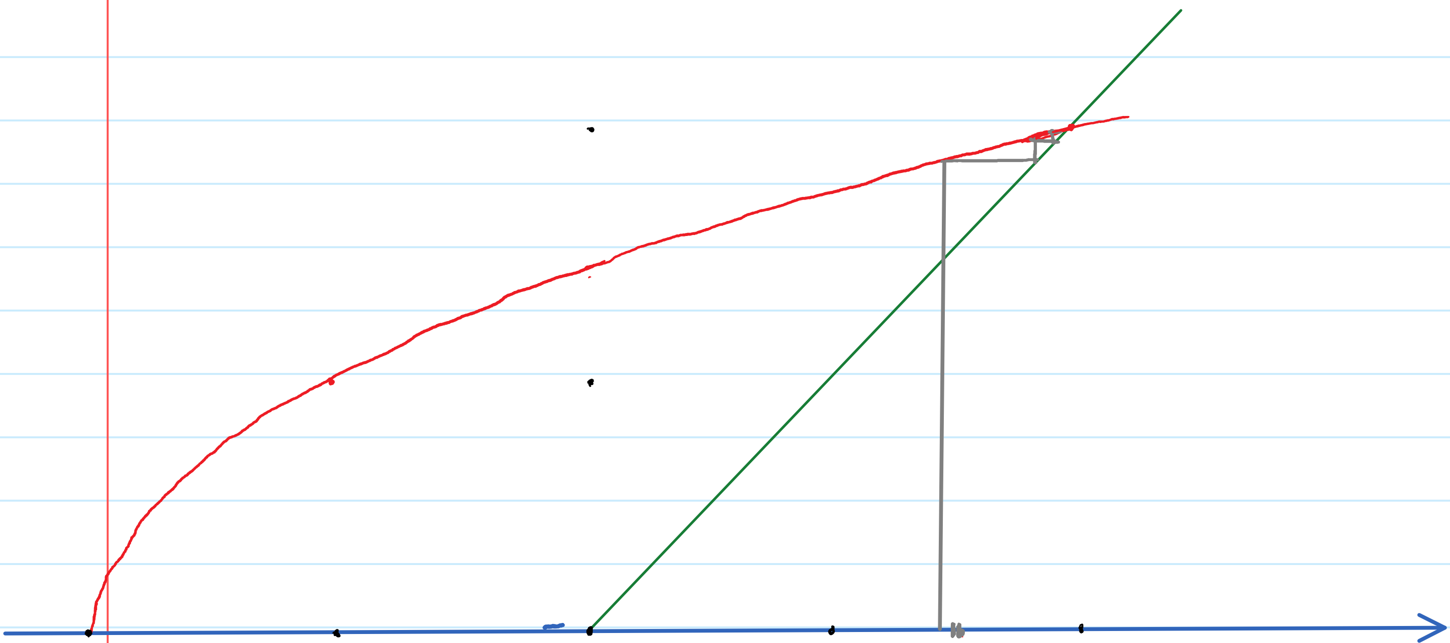


Drawing a segment parallel to the x-axis from $P_1 = (a_1, a_2)$

Lecture 04: Induction principle

Tuesday, August 9, 2022 3:13 PM

till hitting the line $y=x$, we end up getting to the point $Q_1=(a_2,a_2)$. Now going parallel to the y-axis from $Q_1=(a_2,a_2)$ till hitting the graph $y=f(x)$, we end up getting to the point $P_2=(a_2,f(a_2))=(a_2,a_3)$. And we can continue like



From this picture we can "conjecture" that the points (a_n,a_{n+1}) are getting closer and closer to the point of intersection of $y=\sqrt{2+x}$ and $y=x$.

What is this point?

$$\sqrt{2+x}=x \implies 2+x=x^2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0$$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = -1.$$

Since $x \geq 0$, we get that $x = 2$.

To make these arguments formal, we need the induction principle.

> To prove that For every integer $n \geq a$, $P(n)$ holds,
>
> it is enough to show
>
> (Base of induction) $P(a)$ holds
>
> (The inductive step) Suppose for an integer k, $\underbrace{P(k) \text{ holds}}_{\text{(induction hypothesis)}}$.
>
> Then $P(k+1)$ holds.

Using the induction principle, let's prove the first question.

<u>Problem</u>. For a positive integer n, find $1+3+\cdots+(2n-1)$.

We use the sigma notation to show this type of summation.

$$1+3+\cdots+(2n-1) = \sum_{i=1}^{n} (2i-1)$$

i is an auxilary variable ranging from 1 to n. The i-th term is $2i-1$.

<u>Solution</u>. We have already made a conjecture that $1+3+\cdots+(2n-1) = n^2$.

Lecture 4: Induction principle

Friday, October 7, 2016 9:06 AM

We use induction on n to prove

$$\sum_{i=1}^{n} (2i-1) = n^2 .$$

The base case. For $n=1$, the left hand side is 1, the right hand side is 1^2, and $1=1^2$.

The induction step. Suppose for a positive integer k,

$$\sum_{i=1}^{k} (2i-1) = k^2 . \qquad \text{(the induction hypothesis)}$$

We have to prove $\displaystyle\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$.

$$\sum_{i=1}^{k+1} (2i-1) = \underbrace{\sum_{i=1}^{k} (2i-1)} + (2(k+1)-1)$$

$$= k^2 + 2k+1 \qquad \text{(by the induction hypothesis)}$$

$$= (k+1)^2 . \qquad \blacksquare$$

(To clarify let's rewrite the last part without the sigma notation.

Induction hypothesis: for an integer k, $1+3+5+\cdots+(2k-1) = k^2$. Then

$$\underbrace{1+3+\cdots+(2k-1)}_{k^2} + (2k+1) = k^2 + 2k+1 = (k+1)^2 .$$

)

Next we work on understanding $\sqrt{2+\sqrt{2+\cdots}}$. Let's recall

Lecture 4: Induction principle.

Friday, October 7, 2016 9:19 AM

the sequence $a_1=\sqrt{2}$, $a_{n+1}=f(a_n)$ where $f(x)=\sqrt{2+x}$.

Based on our visualization, we conjectured that

1. For every positive integer n, $a_n < a_{n+1}$
2. For every positive integer n, $0 < a_n < 2$

We prove these statements using induction on n.

We proceed with the induction under the assumption that $f(x)$ is increasing; that means $a<b \Rightarrow f(a)<f(b)$.

We will show why f is increasing later.

Base of induction. We have to show $a_1<a_2$ and $0<a_1<2$.

$a_1=\sqrt{2}$, $a_2=\sqrt{2+\sqrt{2}}$. Now we show the desired inequalites using a backward argument:

- $\sqrt{2} < \sqrt{2+\sqrt{2}} \Leftarrow 2 < 2+\sqrt{2} \Leftarrow 0 < \sqrt{2}$.
- $\sqrt{2} < 2 \Leftarrow 2 < 4 \Leftarrow 0 < 2$
- $0 < \sqrt{2}$

Induction step. Suppose for a positive integer k, $a_k < a_{k+1}$ and $0 < a_k < 2$. We have to prove that $a_{k+1} < a_{k+2}$ and $0 < a_{k+1} < 2$.

Lecture 4: Induction principle

Friday, October 7, 2016 9:32 AM

By the induction hypothesis, $a_k < a_{k+1}$. Since f is increasing, we obtain $f(a_k) < f(a_{k+1})$. Notice that $a_{k+1} = f(a_k)$ and $a_{k+2} = f(a_{k+1})$, and so $a_{k+1} < a_{k+2}$.

By the induction hypothesis, $0 < a_k < 2$. Since f is increasing, we deduce that $f(0) < f(a_k) < f(2)$. Notice that $a_{k+1} = f(a_k)$, $f(0) = \sqrt{2} > 0$, and $f(2) = \sqrt{2+2} = 2$. Hence

$$0 < a_{k+1} < 2. \qquad \blacksquare$$

By the previous results, we have that a_n's are increasing, and have an upper bound: $0 < a_1 < a_2 < \cdots < 2$. Therefore $\lim_{n\to\infty} a_n$ exists. Say $\lim_{n\to\infty} a_n = L$. Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L) = \sqrt{2+L}.$$

f is continuous

$\lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+L}$.

Therefore $L^2 - L - 2 = 0$, which implies that $L = 2$ or $L = -1$.

Since $a_n > 0$, $L \geq 0$. Thus $L = 2$, which means $2 = \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$. $\blacksquare$

Lecture 4: Induction principle

Friday, October 7, 2016 12:49 PM

$$\sqrt{2} < \sqrt{2+\sqrt{2}} \Leftarrow 2 < 2+\sqrt{2} \Leftarrow 0 < \sqrt{2} \quad \checkmark.$$

The inductive step. For a give positive integer, we assume $a_k < a_{k+1}$. We have to show $a_{k+1} < a_{k+2}$.

We use backward argument:

$$a_{k+1} < a_{k+2} \Leftarrow \sqrt{2+a_k} < \sqrt{2+a_{k+1}}$$

$$\Leftarrow 2+a_k < 2+a_{k+1}$$

$$\Leftarrow a_k < a_{k+1}$$

which is the induction hypothesis. ■

Theorem. $\lim\limits_{n\to\infty} a_n = 2$, which implies $2 = \sqrt{2+\sqrt{2+\sqrt{\cdots}}}$.

Proof. By Lemma 1, a_n is a bounded sequence.

By Lemma 2, a_n is increasing. Hence $\lim\limits_{n\to\infty} a_n$ exists.

Let $L = \lim\limits_{n\to\infty} a_n$. Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+L}$$

Hence $L^2 = 2+L$ which implies $L^2-L-2=0$. Therefore $(L-2)(L+1)=0$. So $L=2$ or $L=-1$. Since $a_n>0$, $L\geq 0$. Therefore $\lim\limits_{n\to\infty} a_n = 2$. ■