Lecture 10: Limit does not exist.

Sunday, October 30, 2016 10:56 AM

In the previous lecture, we discussed that $\lim_{x\to a} f(x)$ does not exist precisely

when $\forall L\in\mathbb{R},\ \exists \varepsilon>0,\ \forall \delta>0,\ \exists x,\ 0<|x-a|<\delta \wedge |f(x)-L|\geq\varepsilon$

$\forall L\in\mathbb{R},\ \lim_{x\to a} f(x)\neq L$. For some $\varepsilon>0$, there are values x that are arbitrarily close to a and at the same time $f(x)$ is ε-away from L.

To prove a limit does not exist we often use the following strategy.

(a) Find two sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ with the following properties.

(1) $x_n\to a$ as $n\to\infty$ and $y_n\to a$ as $n\to\infty$.

(x_n's and y_n's get closer and closer to a as n gets larger and larger)

(2) $f(x_n)\to L_1$ as $n\to\infty$ and $f(y_n)\to L_2$ as $n\to\infty$ for two numbers $L_1\neq L_2$.

(b) Suppose to the contrary that $\lim_{x\to a} f(x)=L$. Let ε be a positive number less than $\frac{|L_1-L_2|}{2}$. Use x_n's and y_n's, to show that L is ε-close to L_1 and L_2. Get a contradiction.

Here we are going to use this strategy to show $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. This type of argument is common in analysis courses.

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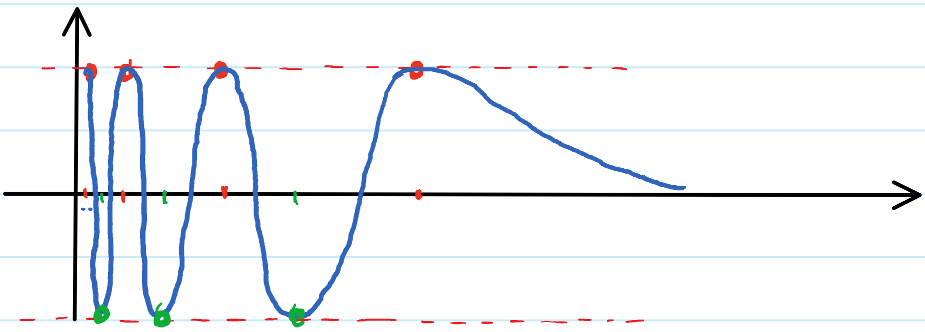
Problem. Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does NOT exist.

First we visualize the problem by looking at the graph

$y = \sin\left(\frac{1}{x}\right)$



As you can see the blue curve can get close to any point on the segment $[-1,1]$ in the y-axis. (The set which consists of the mentioned segment and graph of $\sin(1/x)$ is an interesting set. In topology you will learn that this set is connected, but it is not path-connected.)

We focus on the points at "top" and "bottom". I.e. we will find two sequences x_n and y_n with the following properties both x_n and y_n get closer and closer to zero; for every n, $\sin(1/x_n^+) = 1$ and $\sin(1/x_n^-) = -1$.

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Let's see how having these sequences is sufficient to deduce that

$\lim_{x \to 0} \sin(1/x)$ does not exist.

Assume to the contrary that $\lim_{x\to 0} \sin(1/x) = L$. So

for every $\varepsilon > 0$, there exists $\delta > 0$ such that

if x is δ-close to 0, then $\sin(1/x)$ is ε-close to L.

Since x_n, y_n are getting closer and closer to 0, eventually they

get δ-close to 0. Hence $\sin(\frac{1}{x_n}), \sin(\frac{1}{y_n})$ are ε-close to L.

Therefore, both 1 and -1 are ε-close to L. Thus L is

ε-close to 1 and ε-close to -1. But, for $\varepsilon < 1$, there is

no number which is both ε-close to 1 and ε-close to -1.

This gives us a contradiction.

Here is the formal proof:

Step 1. There exists a sequence x_n of numbers such that

(a) x_n gets closer and closer to 0. I.e.

$\forall \delta > 0, \exists N \in \mathbb{R}, n > N \Rightarrow |x_n| < \delta$.

(b) $\sin(1/x_n) = 1$.

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Proof of Step 1. We start with part (b) and use a backward argument:

$$\sin(1/x_n) = 1 \Leftarrow \frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$$

$$\Leftarrow x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}.$$

To get part (a), we start with a given $\delta > 0$ and again use backward argument to find a suitable N.

$$|x_n| < \delta \Leftarrow \frac{1}{\frac{\pi}{2} + 2n\pi} < \delta$$

$$\Leftarrow \frac{1}{2n\pi} < \delta$$

$$\Leftarrow \frac{1}{2\pi\delta} < n.$$

($N = \frac{1}{2\pi\delta}$ is a suitable choice.)

Step 2. There exists a sequence y_n such that

(a) y_n gets closer and closer to 0.

$\forall \delta > 0, \exists N > 0, n \geq N \Rightarrow |y_n| < \delta$.

(b) $\sin(1/y_n) = -1$.

Proof of step 2. It is similar to the proof of Step 1.

$$\sin(1/y_n) = -1 \Leftarrow \frac{1}{y_n} = -\frac{\pi}{2} + 2n\pi \Leftarrow y_n = \frac{1}{-\frac{\pi}{2} + 2n\pi}.$$

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$$|y_n| < \delta \Longleftarrow \frac{1}{-\frac{\pi}{2}+2n\pi} < \delta$$

$$\Longleftarrow \frac{1}{2(n-1)\pi} < \delta$$

$$\Longleftarrow \frac{1}{2\pi\delta} < n-1 \Longleftarrow \frac{1}{2\pi\delta} + 1 < n$$

(So, for $\delta>0$, $N = \dfrac{1}{2\pi\delta} + 1$ is a suitable choice.)

<u>Finishing the proof.</u> Suppose to the contrary $\lim_{x\to 0} \sin(1/x) = L$.

In particular, there is $\delta_0 > 0$ such that

if $0<|x|<\delta_0$, then $\sin(1/x)$ is $\frac{1}{2}$-close to L. (I)

By Step 1 and Step 2, there is N such that

$$n \geq N \Rightarrow (0 < |x_n| < \delta_0 \text{ and } 0 < |y_n| < \delta_0) \quad \text{(II)}$$

Hence, by (I), (II),

$n \geq N \Rightarrow$ $\sin\left(\frac{1}{x_n}\right)$ and $\sin\left(\frac{1}{y_n}\right)$ are $\frac{1}{2}$-close to L

$$\Rightarrow \left|\sin\left(\frac{1}{x_n}\right) - L\right| < \frac{1}{2} \text{ and } \left|\sin\left(\frac{1}{y_n}\right) - L\right| < \frac{1}{2}$$

$$\Rightarrow |1-L| < \frac{1}{2} \text{ and } |-1-L| < \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} < L < \frac{3}{2} \text{ and } -3/2 < L < -1/2$$

which is a contradiction. ∎

Lecture 10: Limit does not exist

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Problem. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \quad \text{(rational)} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \quad \text{(irrational)}. \end{cases}$

Prove that, for any $a \in \mathbb{R}$, $\lim_{x \to a} f(x)$ does NOT exist.

Sketch of a proof. To show this it is enough to notice that every real number a can be approximated by rational numbers x_n and irrational numbers y_n. So

- $\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} y_n = a$.
- $f(x_n) = 1$ and $f(y_n) = 0$ for every n.

Hence by the above mentioned method, one can show that $\lim_{x \to a} f(x)$ does NOT exist.

Lecture 10: Cartesian product

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René Descarte used coordinates to study geometry. Nowadays we use the idea of <u>n-tuples</u> in many aspects of our life:

<u>Ex.</u> <u>List of courses</u>: it has various columns; name, number, location, ...

<u>List of movies in netflix</u>: genre, title, length, rating, etc.

<u>Definition</u>. Given sets X and Y, the <u>Cartesian product</u> of X and Y, denoted by $X \times Y$, is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\},$$

where (x, y) is an ordered-pair, i.e. $(x_1, y_1) = (x_2, y_2)$ exactly when $x_1 = x_2$ and $y_1 = y_2$.

Similarly we define $X_1 \times X_2 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i \text{ for } 1 \le i \le n\}$, and $(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$ if and only if $x_i = x'_i$ for $1 \le i \le n$.

<u>Ex.</u> Let $A = \{1, 2\}$ and $B = \{a, b\}$. List elements of $A \times B$, and $B \times A$.

<u>Solution</u>. $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Lecture 10: Cartesian product

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We pair each element of A by all the elements of B.

In the above example, you can see that $(A \times B) \cap (B \times A) = \emptyset$.

<u>Ex.</u> Let $A = \{1, 2\}$ and $B = \{1, 3, 4\}$. Find $(A \times B) \cap (B \times A)$.

<u>Solution</u>

$$A \times B = \{(1,1), (1,3), (1,4), (2,1), (2,3), (2,4)\}$$

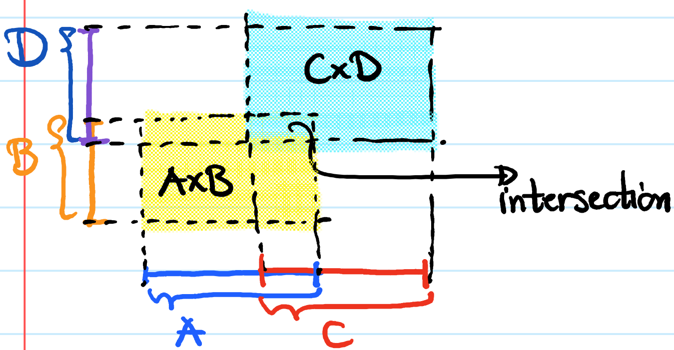
$$B \times A = \{(1,1), (1,2), (3,1), (3,2), (4,1), (4,2)\}$$

$$(A \times B) \cap (B \times A) = \{(1,1)\}. \quad \blacksquare$$

<u>Lemma</u>. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

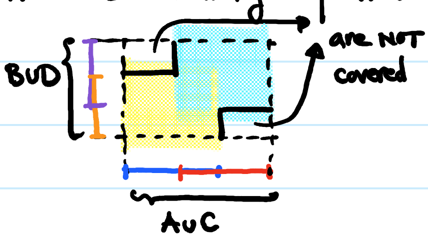
<u>Proof</u>.

$$\begin{aligned}
(x,y) \in (A \times B) \cap (C \times D) &\iff (x,y) \in A \times B \wedge (x,y) \in C \times D \\
&\iff x \in A \wedge y \in B \wedge x \in C \wedge y \in D \\
&\iff (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\
&\iff x \in A \cap C \wedge y \in B \cap D \\
&\iff (x,y) \in (A \cap C) \times (B \cap D). \quad \blacksquare
\end{aligned}$$



<u>Warning</u>. $(A \times B) \cup (C \times D)$ is not necessarily equal to $(A \cup C) \times (B \cup D)$.

(why ?)



Lecture 10: Cartesian product and counting

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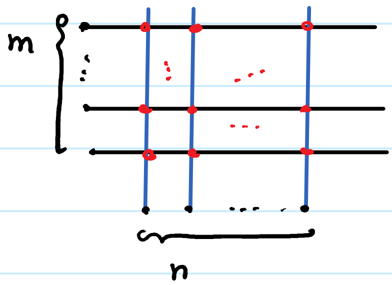
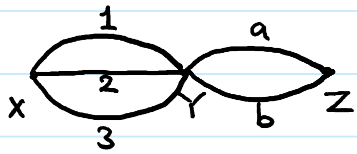
Based on your intuition of cardinality of finite sets, you can see that $|A \times B| = |A|\,|B|$ if A and B are finite sets.



Ex. In the following pictures in how many ways can we go from X to Z by passing Y only once.

Solution. We can "label" each path with an element of $\{1,2,3\} \times \{a,b\}$. And any element of $\{1,2,3\} \times \{a,b\}$ is a label of a path. So there is a "matching" (the technical term is bijection as we will learn later) between the possible paths and elements of $\{1,2,3\} \times \{a,b\}$



So there are 6 possible paths. ∎

The key point in the above example is the following:

We often count objects by finding a bijection between them and a more familiar set. A set whose cardinality is already known.

Lecture 10: Functions

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"Definition" A function carries three pieces of information:

. Two sets: one is called domain and the other is called codomain.

. A rule: assigns a unique element of codomain to each element of domain

We either write $f: X \to Y$ and then specify its rule,

or $X \xrightarrow{f} Y$

$x \mapsto f(x)$

. You have worked with a lot of functions in calculus, but in an inaccurate way. In the following examples we will see some of these inaccuracies.

Ex. Is the following a function?

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 1/x$.

Answer. No, f is NOT defined 0. ■

By changing its domain, we can address this issue:

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = 1/x$ is a function.

Lecture 10: Function, composition

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<u>Ex.</u> Is the following a function?

$$f: \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2.$$

<u>Answer</u>. No, it is NOT. It assigns 0 to 0 which does NOT belong to the codomain $\mathbb{R}^+$. ∎

By changing the codomain we can address this issue:

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \quad \text{is a function.}$$

<u>Ex.</u> Is the following a function?

$$f: \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = y \text{ if } y^2 = x.$$

<u>Answer</u>. No, it is NOT. This rule does NOT assign a <u>unique</u> element of codomain to, let's say, 1. We have $(\pm 1)^2 = 1$. ∎

Changing the codomain can resolve this issue:

$$\left(f: \mathbb{R}^+ \to \mathbb{R}^+, \; f(x) = y \text{ if } y^2 = x \right) \text{ is a function.}$$

In fact, in this case, $f: \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = \sqrt{x}$.

<u>Composition of functions</u> Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be two functions; suppose codomain of f is equal to the domain

of g. Then we can form a new function called the composition of f and g, denoted by $g \circ f$.

$$\begin{tikzcd} x \arrow[r, mapsto] & f(x) \arrow[d, mapsto] \\ & g(f(x)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

$$(g\circ f)(x) = g(f(x))$$

Domain of $g\circ f$ = Domain of f

Codomain of $g\circ f$ = codomain of g

Rule of $g\circ f$: $x \mapsto g(f(x))$.

Ex. Let $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = 1/x$. Find $f\circ f$.

Answer. It does NOT make sense to talk about $f\circ f$ the codomain of f is NOT equal to the domain of f. ■

This issue can be resolved by changing the codomain of f.

Let $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $f(x) = 1/x$. Then

$f\circ f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $(f\circ f)(x) = f(f(x))$

$$= \frac{1}{f(x)} = \frac{1}{1/x} = x.$$

Remark. $f\circ f$ is not equal to $I: \mathbb{R} \to \mathbb{R}$, $I(x) = x$ as they have different (co) domains.