Name: ____________________

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 10 | |
| 4 | 10 | |
| Total: | 30 | |

1. (5 points) Suppose $G$ is a finite $p$-group and $1 \neq N \trianglelefteq G$. Prove that

$$N \cap Z(G) \neq \{1\}.$$

2. (5 points) Suppose $G$ is generated by $d$ elements. Prove that

$$|\{H \leq G| \ [G:H] \leq n\}| \leq (n!)^d.$$

3. (10 points) Suppose $G$ is a finite group, $H \trianglelefteq G$, and $p$ is a prime factor of $|H|$. Prove that $|\mathrm{Syl}_p(H)|$ divides $|\mathrm{Syl}_p(G)|$, where $\mathrm{Syl}_p(G)$ (resp. $\mathrm{Syl}_p(H)$) is the set of Sylow $p$-subgroups of $G$ (resp. $H$).

4. (10 points) Classify groups of order 306 that have a cyclic 3-subgroup. (Hint: $4 \nmid 306$.)