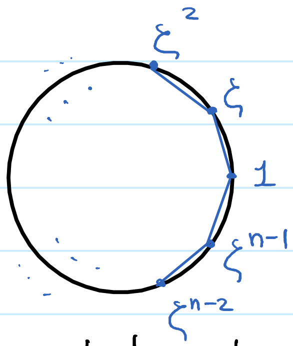
Homework 1

Wednesday, October 4, 2017 8:36 AM

1. Let S_n be the symmetric group of $\{1,2,\ldots,n\}$. For any $\sigma\in S_n$, let $m_\sigma := |\{ i\in\{1,2,\ldots,n\} \mid \sigma(i)=i\}|$. Find $\sum_{\sigma\in S_n} m_\sigma$.

2. Let P_n be the regular n-gon with vertices

$$1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$$



where $\zeta = e^{2\pi i/n}$.

Let $\tau: P_n \to P_n$ be the restriction of the rotation by angle $\frac{2\pi}{n}$ around the origin; and $\sigma: P_n\to P_n$ be the restriction of the reflection about the x-axis. Let D_{2n} be the combinatorial symmetries of P_n.

- (Rigidity) Convince yourself that, if $g_1, g_2\in D_{2n}$ and $g_1(1)=g_2(1)$ and $g_1(\zeta)=g_2(\zeta)$, then $g_1=g_2$.
- Prove that, if $g\in D_{2n}$, then either $g=\tau^i$ or $g=\sigma\tau^i$ for some $0\le i\le n-1$. And so

$$D_{2n}=\{1,\tau,\ldots,\tau^{n-1},\sigma,\sigma\tau,\ldots,\sigma\tau^{n-1}\}.$$

- Prove that $\sigma\tau\sigma^{-1}=\tau^{-1}$.

Homework 1

Wednesday, October 4, 2017 10:30 AM

3. In class, we will prove that, if G is a finite group and H is a proper subgroup, then $G \neq \bigcup_{g \in G} gHg^{-1}$. Is this true for infinite groups?

4. Let $SL_2(\mathbb{R})$ be the set real 2x2 matrices with determinant 1. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{C}$, let

$$\circledast \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}.$$

ⓐ Prove that $\circledast$ defines a group action $SL_2(\mathbb{R}) \curvearrowright \mathbb{C}$.

ⓑ Convince yourself that $\operatorname{Im}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z\right) = \frac{\operatorname{Im}(z)}{|cz+d|^2}$.

Prove that $SL_2(\mathbb{R})$ has three orbits: the upper half plane $\mathcal{H}$, the real axis, and the lower half plane $\mathcal{H}^-$.

ⓒ Show that the stabilizer of i is the special orthogonal group $SO_2(\mathbb{R}) := \{ g \in SL_2(\mathbb{R}) \mid gg^t = I \}$.

5. Recall that a group G is called simple if the only normal subgps of G are $\{e\}$ and G. Suppose G is a simple group and H is a proper subgroup of index n. Prove that G can be embedded into S_n. (Hint. Consider $G \curvearrowright G/H$.)

Homework 1

Wednesday, October 4, 2017 11:05 AM

6. Let G be a group, and X be a finite set.

Let $L^2(X) := \{ f: X \to \mathbb{C} \mid f \text{ is any function} \}$, and

$$\langle f_1, f_2 \rangle := \sum_{x \in X} f_1(x) \overline{f_2(x)} .$$

Convince yourself that $L^2(X)$ is just the vector space $\mathbb{C}^{|X|}$

(list elements $x_1, \ldots, x_n$ of X and think about

$$L^2(X) \longrightarrow \mathbb{C}^{|X|}$$
$$f \longmapsto (f(x_1), \ldots, f(x_n)) .\ )$$

Suppose $G \curvearrowright X$.

(a) Prove that the following defines an action $G \curvearrowright L^2(X)$:

$$(g * f)(x) := f(g^{-1} \cdot x) .$$

(b) Prove that, $\forall f_1, f_2 \in L^2(X), \forall g \in G, \langle g * f_1, g * f_2 \rangle = \langle f_1, f_2 \rangle$

(We say it is a <u>unitary action</u>.)

(c) Convince yourself that, $\forall g \in G$,

$$\lambda_g : L^2(X) \to L^2(X),\ \lambda_g(f) := g * f$$

is a linear map. Prove that

$$tr(\lambda_g) = \#\text{ of the fixed points of } g$$

(that means $|\{x \in X \mid g \cdot x = x\}|$).

(<u>Hint</u>. Use the following basis for $L^2(X)$: $\{\delta_x\}_{x \in X}$ where

$$\delta_x : X \to \mathbb{C},\ \delta_x(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{if } x \neq x' . \end{cases})$$

Homework 1

Wednesday, October 4, 2017 11:40 AM

7. Suppose G is a finite group, $C \subseteq \mathbb{R}^n$ is a convex subset; that means, if $p, q \in C$, then the segment pq is in C. Suppose $G \curvearrowright C$ by affine actions; that means $\forall p, q \in C$, $\forall t \in [0,1]$, $\forall g \in G$,

$$g \cdot (tp + (1-t)q) = t\, g \cdot p + (1-t)\, g \cdot q.$$

Prove that G has a fixed point; that means

$$\exists\, x \in C \text{ s.t. } \forall g \in G,\ g \cdot x = x.$$

(Hint. ① Suppose $c_1, \dots, c_n \in C$. By the convexity of C, using induction show the average $\frac{1}{n}(c_1 + c_2 + \cdots + c_n)$ is in C.

② Take $y \in C$, and let x be the average of the G-orbit of y. Prove that x is a fixed point of G.)

8. Suppose G is a finite subgroup of the group $GL_n(\mathbb{R})$ of $n \times n$ real invertible matrices. Prove that there is an inner product on $\mathbb{R}^n$ which is G-invariant.

(Recall. $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called an inner product if

ⓐ $\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$

ⓑ $\langle v, c_1 w_1 + c_2 w_2 \rangle = c_1 \langle v, w_1 \rangle + c_2 \langle v, w_2 \rangle$

ⓒ $\langle v, v \rangle > 0$ if $v \neq 0$.

ⓓ $\langle v, w \rangle = \langle w, v \rangle$

Homework 1

Wednesday, October 4, 2017 11:58 AM

for instance $(a_1,\ldots,a_n)\bullet(b_1,\ldots,b_n) = a_1b_1+a_2b_2+\cdots+a_nb_n$

is an inner product.)

(Hint. Define $\langle v,w\rangle := \frac{1}{|G|}\sum_{g\in G} gv\bullet gw$

(the average of the standard inner product along the

G-orbits of v and w.); you have to show $\langle\, ,\, \rangle$ is

an inner product and $\langle gv, gw\rangle = \langle v,w\rangle$.)

[This problem is extremely useful as it implies:

if $V\subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ which is invariant

under G (that means $\forall v\in V, \forall g\in G$, we have $g\cdot v\in V$.)

then $V^{\perp} := \{w\in\mathbb{R}^n \mid \forall v\in V, \langle w,v\rangle = 0\}$ is

also G-invariant, and $V\oplus V^{\perp} = \mathbb{R}^n$.]

9. In class, we recalled that $c: G\longrightarrow \mathrm{Aut}(G)$, $c(g)=c_g$ where

$c_g(g') = g g' g^{-1}$ is a group homomorphism, the image of c

is called the group of inner automorphisms of G, and it is denoted

by $\mathrm{Inn}(G)$. (a) Prove that $\ker(c)$ is the center $Z(G)$ of G.

Homework 1

Wednesday, October 4, 2017 7:52 PM

(b) Deduce that $\mathrm{Inn}(G) \simeq G/Z(G)$.

(c) Prove that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

(d) Prove that $|Z(\mathrm{Aut}(G))| \leq |\mathrm{Hom}(G, Z(G))|$; in particular, if either $Z(G)=1$ or G is perfect (that means $G=[G,G]$), then $Z(\mathrm{Aut}(G))$ is trivial.

(Hint ① $\forall g\in G$ and $\forall \phi\in \mathrm{Aut}(G)$, $\phi\circ c_g\circ\phi^{-1} = c_{\phi(g)}$;

② If $\phi\in Z(\mathrm{Aut}(G))$, then $c_g = c_{\phi(g)}$; and so $\phi(g) = g\,\eta(g)$ for some $\eta(g)\in Z(G)$.

③ Prove $\eta\in \mathrm{Hom}(G, Z(G))$.)

10. Recall that we say $G \curvearrowright X$ transitively if $|G\backslash X|=1$.

A transitive group action $G\curvearrowright X$ is called <u>primitive</u> if it does not preserve any non-trivial partition of X, where trivial partitions are $\{X\}$ and $\{\{x\} \mid x\in X\}$.

For instance, let $\sigma: \{1,2,3,4\} \to \{1,2,3,4\}$, $1 \xmapsto{\sigma} 2 \xmapsto{\sigma} 3 \xmapsto{\sigma} 4 \xmapsto{\sigma} 1$. Then $\{\{1,3\},\{2,4\}\}$ is

Homework 1

Wednesday, October 4, 2017 11:07 PM

preserved by $\langle\sigma\rangle$; so $\langle\sigma\rangle \curvearrowright \{1,2,3,4\}$ is NOT primitive though it is transitive.

Suppose $G \curvearrowright X$ is a non-trivial transitive Then

$G \curvearrowright X$ is primitive if and only if for any $x \in X$ the stabilizer group G_x of x is a <u>maximal</u> subgroup; that means ① G_x is a proper subgp

② $G_x \leq H \leq G \Rightarrow$ either $G_x = H$ or $G = H$.

(<u>Hint</u>• Since $G \curvearrowright X$ is transitive, $X = G \cdot x$;

If $\exists\, G_x \lneqq H \lneqq G$, then show that $\{ gH \cdot x \mid g \in G \}$ is a non-trivial partition of X which is preserved by $G \curvearrowright X$.

• Suppose $\{ X_i \mid i \in I \}$ is a partition which is preserved by the G-action. So $\forall g$, $g \cdot X_i = X_{\sigma_g(i)}$ where $\sigma_g \in S_I$.

Suppose $|X_0| \geq 2$; and $x \in X_0$.

$\forall g \in G_x$, $g \cdot X_0 \cap X_0 \neq \emptyset$, which implies $gX_0 = X_0$.

So $G_{X_0} \supseteq G_x$. Since $|X_0| \geq 2$ and $G \curvearrowright X$ is transitive, $G_{X_0} \supsetneqq G_x$. Since $\exists\, x' \in X \setminus X_0$ and $G \curvearrowright X$ is trans. $G_{X_0} \subsetneqq G$.)