Homework 6

Wednesday, November 15, 2017 11:44 AM

1. Suppose G is a finite group and $H \leq G$.

(a) Prove that $H = G \iff H\Phi(G) = G$.

(b) Let $\pi : G \to G/\Phi(G)$ be the natural projection map. Suppose $S \subseteq G$. Prove that $\langle S \rangle = G \iff \langle \pi(S) \rangle = G/\Phi(G)$.

In particular $\langle S \rangle = G \iff \langle S \setminus \Phi(G) \rangle = G$.

(c) Let $d(G) :=$ the min. number of generators of G.

Prove that $d(G) = d(G/\Phi(G))$.

[Hint (a) If $H \neq G$, then $\exists$ a max. subgp $H \leq M < G$.

$\Rightarrow M/\Phi(G) \neq G/\Phi(G)$.]

2. Suppose G is a finite group; and $\Phi(G)$ is the Frattini subgroup of G.

(a) Suppose P is a Sylow subgroup of $\Phi(G)$. Prove that

$$P \triangleleft G.$$

(b) Prove that $\Phi(G)$ is nilpotent.

[Hint. (a) Use Frattini's argument: $N_G(P)\Phi(G) = G$, and deduce $N_G(P) = G$.]

3. Suppose G is a finite p-group; and $d(G)$ is the min. number of generators of G.

(a) Prove that $d(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} (G/G^p[G,G])$.

Homework 6

Wednesday, November 15, 2017 12:12 PM

(b) Suppose S is a <u>minimal</u> generating set of G; that means $\langle S\rangle = G$ and $\langle S'\rangle \neq G$ if $S' \subsetneq S$.

Prove that $|S| = d(G)$.

(c) Does part (b) hold for finite groups that are not p-groups; that means for a finite group H do we have $|S_1| = |S_2|$ if S_1 and S_2 are two minimal generating sets?

[Hint (c) $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.]

4. Prove that, if $G/Z(G)$ is nilpotent, then G is nilpotent.

5. (a) Prove that $G/Z(G)$ cannot be a non-trivial cyclic group.

(b) Prove that any group of order p^2 is abelian.

(c) Suppose G is a non-abelian group of order p^3. Prove that

(c1) $Z(G) \simeq \mathbb{Z}/p\mathbb{Z}$.

(c2) $Z(G) = [G,G]$, and $G/Z(G) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(c3) $d(G)=2$; that means G can be gen. by 2 elements, but not by 1 element!

Homework 6

Wednesday, November 15, 2017 12:52 PM

6. Let $G := GL_n(\mathbb{Z}/p\mathbb{Z})$ be the set of $n\times n$ invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. Let V be the n-dimensional vector space $\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$. Let

$$X := \{(v_1, \ldots, v_n) \mid v_1 \neq 0;\ v_2 \notin \langle v_1\rangle;\ v_3 \notin \langle v_1, v_2\rangle;\ \ldots;\ v_n \notin \langle v_1, \ldots, v_{n-1}\rangle\}.$$

For any $g \in GL_n(\mathbb{Z}/p\mathbb{Z})$ and $(v_1, \ldots, v_n) \in X$, let

$$g \cdot (v_1, \ldots, v_n) := (gv_1, \ldots, gv_n). \quad (*)$$

Convince yourself that $(*)$ defines a group action $G \curvearrowright X$.

(a) Prove that $G \curvearrowright X$ transitively.

(b) Prove that $G_{(e_1, \ldots, e_n)} = \{I\}$ where $\{e_1, \ldots, e_n\}$ is the standard basis of V.

(c) Prove that
$$|GL_n(\mathbb{Z}/p\mathbb{Z})| = (p^n-1)(p^n-p)\cdots(p^n-p^{n-1})$$
$$= p^{\frac{n(n-1)}{2}}(p^n-1)(p^{n-1}-1)\cdots(p-1).$$

(d) $x \in GL_n(\mathbb{Z}/p\mathbb{Z})$ is called <u>unipotent</u> if $(x-I)^n = 0$.

Suppose $U \leq GL_n(\mathbb{Z}/p\mathbb{Z})$ and $\forall u \in U$ is unipotent. Prove that $\exists g \in G$, $gUg^{-1} \subseteq \left\{\begin{bmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1\end{bmatrix} \;\middle|\; x_{ij} \in \mathbb{Z}/p\mathbb{Z}\right\}$.

(<u>Hint</u> (d), show that $x \in GL_n(\mathbb{Z}/p\mathbb{Z})$ is unipotent $\Longrightarrow$ $o(x)$ is a power of p; and find a Sylow p-subgp.)