Homework 8

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$\boxed{1}$ Suppose R is a unital ring that is not necessarily commutative. Prove that $\tilde{I}$ is a both sided ideal of $M_n(R)$ if and only if $\tilde{I} = M_n(I)$ for some both sided ideal I of R.

(Remark. Proof of this result just needs a bit of patience with matrix computations, but it is an important result. In particular, it shows that $M_n(\mathbb{C})$ has no proper non-zero both sideded ideal.)

(Hint. Let $e_{ij} \in M_n(R)$ s.t. the i,j entry of e_{ij} is 1 and the other entries are 0. Then $e_{ij}\, e_{kl} = \begin{cases} 0 & \text{if } j \neq k, \\ e_{il} & \text{if } j = k. \end{cases}$

So, for $a = [a_{ij}]$, $e_{kk}\, a\, e_{ll} = \sum_{i,j} a_{ij}\, e_{kk} e_{ij} e_{ll}$

$= a_{kl}\, e_{kl}$.

- Let $I := \{x \in R \mid x \text{ is an entry of an element of } \tilde{I}\}$.)

$\boxed{2}$ A unital ring R (not necessarily commutative) is called a <u>division ring</u> if $U(R) = R \setminus \{0\}$. (So a commutative ring is a division ring if and only if it is a field.)

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Let $\mathbb{H} := \{ \begin{bmatrix} z & \omega \\ -\bar{\omega} & \bar{z} \end{bmatrix} \in M_2(\mathbb{C}) \}$. Convience yourself that $\mathbb{H}$ is a subring of $M_2(\mathbb{C})$. Prove that $\mathbb{H}$ is a division ring. Show that $\mathbb{H}$ is not commutative.

$\boxed{3}$ Convience yourself that $\mathbb{Q}[\sqrt{2}] = \{ a + \sqrt{2}\, b \mid a, b \in \mathbb{Q} \}$ is a subring of $\mathbb{R}$. Let $\phi : \mathbb{Q}[\sqrt{2}] \longrightarrow M_2(\mathbb{Q})$,

$$\phi(a + \sqrt{2}\, b) := \begin{bmatrix} a & b \\ 2b & a \end{bmatrix}.$$

Prove that ϕ is a ring homomorphism, and deduce that

$$\mathbb{Q}[\sqrt{2}] \simeq \{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} \mid a, b \in \mathbb{Q} \}.$$

$\boxed{4}$ Let I be the ideal generated by 2 and x in $\mathbb{Z}[x]$. Prove that I is not a principal ideal.

$\boxed{5}$ (a) Suppose R is a unital commutative ring. Prove that

$$R[x]^{\times} = \{ \sum_{i=0}^{n} a_i x^i \mid a_0 \in R^{\times},\ a_1, \ldots, a_n \in \mathrm{Nil}(R) \}.$$

(b) Prove that $\mathrm{Nil}(R/_{\mathrm{Nil}(R)}) = 0$.

$\boxed{6}$ Let $\omega := \frac{-1 + i\sqrt{3}}{2}$ and $\mathbb{Z}[\omega] = \{ a + b\omega \mid a, b \in \mathbb{Z} \}$.

(a) Let $N(a + b\omega) := (a + b\omega)(a + b\bar{\omega})$. Convience yourself that

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$N(z_1 z_2) = N(z_1)N(z_2)$ and $N(a+\omega b) = a^2 - ab + b^2$.

Prove that $\forall z_1, z_2 \in \mathbb{Z}[\omega]$, $z_2 \neq 0$, $\exists q, r \in \mathbb{Z}[\omega]$ s.t.

$z_1 = z_2 \cdot q + r$ and $N(r) < N(z_2)$.

(b) Prove that $\mathbb{Z}[\omega]$ is a PID.

(c) Prove that $\mathbb{Z}[\omega]^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$.

__Extra__. Can you use part (b) to show $x^3 + y^3 = z^3$ does not have a non-trivial solution in $\mathbb{Z}$?